The Logical Complexity of Schanuel's Conjecture

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March 12, 2017

Schanuel's Conjecture

Schanuel's Conjecture: If $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, then the transcendece degree of $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, \exp(\lambda_1), \ldots, \exp(\lambda_n))$ is at least n.

Question from FOM: The natural formulation of Schanuel's Conjecture is Π^1_1. Is there an equivalent arithmetic formulation?

After a long digression I will argue

- If Schanuel's Conjecture is false there are recursive counterexamples.
- Schanuel's Conjecture is equivalent to a Π^0_3-sentence.

Exponential Algebraic Closure


We begin with a discussion of *Exponential Algebraic Closure* introduced by Alex Wilkie and developed further by Jonathan Kirby.

Definition

An *exponential field* is a characteristic zero field K and an non-trivial $E : K \to K$ such that $E(a + b) = E(a)E(b)$.

All of our fields will have characteristic zero.

Exponetial Derivations

Definition

A *derivation* on K is a map $D : K \to K$ such that

$$D(a+b) = D(a) + D(b) \text{ and } D(ab) = aD(b) + bD(a)$$

An *exponential derivation* is a derivation $D : K \to K$ such that $D(E(a)) = D(a)E(a)$

Definition

For $C \subset K$ let $\mathrm{EDer}(K/C)$ be the set of exponential derivations on K such that $D(c) = 0$ for $c \in C$.

Define $\mathrm{Der}(K/C)$ similarly.

Exponential Algebraic Closue

The Classical Case: Let K be a field of characteristic 0, $a \in K$, $B \subset K$, then $a \in \mathrm{acl}(B)$ if and only if $D(a) = 0$ for all $D \in Der(K/B)$.

Definition (Exponential Closure)

Let K be an exponential field, $a \in K$ and $C \subset K$. Then $a \in \mathrm{Ecl}(C)$ if and only if $D(a) = 0$ for all $D \in \mathrm{EDer}(K/C)$.

Basic properties:

- $\mathrm{Ecl}(A)$ is an exponential field;
- $A \subseteq \mathrm{Ecl}(A)$;
- $\mathrm{Ecl}(\mathrm{Ecl}(A)) = \mathrm{Ecl}(A)$;
- $A \subseteq B \Rightarrow \mathrm{Ecl}(A) \subseteq \mathrm{Ecl}(B)$.

Exchange

Lemma (Exchange)

If $b \in \mathrm{Ecl}(A, c)$*, then* $b \in \mathrm{Ecl}(A)$ *or* $c \in \mathrm{Ecl}(A, b)$

Proof Suppose $b \in \mathrm{Ecl}(A, c)$ but $c \notin \mathrm{Ecl}(A, b)$. Let $D \in \mathrm{EDer}(K/A)$. Want $D(b) = 0$ so $b \in \mathrm{Ecl}(A)$.

Since $c \notin \mathrm{Ecl}(A, b)$, there is $D_1 \in \mathrm{Der}(K/A)$ with $D_1(c) = 1$ and $D_1(b) = 0$.

Let $D_2 = D - D(c)D_1 \in \mathrm{EDer}(K/A)$.
Then $D_2(b) = D(b)$ and $D_2(c) = D(c) - D(c) = 0$. '

Thus, since $b \in \mathrm{Ecl}(A, c)$, $D(b) = D_2(b) = 0$ and $b \in \mathrm{Ecl}(A)$.

Finite Character

Lemma (Finite Character)

If $b \in \mathrm{Ecl}(A)$, then there is $A_0 \subseteq A$ finite such that $b \in \mathrm{Ecl}(A_0)$.

Sketch of Proof

Corollary

a) Ecl *is a pregeometry;*
b) Any two bases for $\mathrm{Ecl}(A)$ *have the same cardinality, which we will call* $\mathrm{Edim}(A)$.

Exponential Closure and Transcendence

Theorem (Ax)

Let K be a field and $\Delta \subset \mathrm{Der}(K)$. Let $C = \bigcap_{D \in \Delta} ker(D)$ and suppose $x_1, \ldots, x_n, y_1, \ldots, y_n \in K$ such that $Dy_i = y_i Dx_i$ for all i and $D \in \Delta$. Then

$$td(\overline{x}, \overline{y}/C) - ldim_{\mathbb{Q}}(\overline{x}/C) \geq \ rank\ (Dx_i)_{D \in \Delta, i=1,\ldots,n}$$

where $ldim_Q$ is the $\mathbb{Q}$-linear dimension of $\overline{x}$ over C.

Let K be an exponential field and let $C \subset K$ such that $C = \mathrm{Ecl}(C) = \bigcap_{D \in EDer(K/C)} ker(D)$.
Let $x_1, \ldots, x_n \in K$ and $y_i = E(x_i)$.
Suppose $m = \mathrm{Edim}(\overline{x}/C)$ and wlog $x_1, \ldots, x_m$ is an Ecl-basis over C.
Choose D_i such that $D_i(x_i) = 1$ and $D_i(x_j) = 0$ for $i, j \leq m$ and $i \neq j$.
Then rank $(D_i(x_j)) = m$.
Thus $td(\overline{x}, E(\overline{x})/C) - ldim_{\mathbb{Q}}(\overline{x}/C) \geq \mathrm{Edim}(\overline{x}/C)$

Essential Counterexample to Schanuel's Conjecture

Let $\delta(\overline{a}/B) = td(\overline{a}, E(\overline{a})/B, E(B)) - ldim_{\mathbb{Q}}(\overline{a}/B)$
and $\delta(\overline{a}) = \delta(\overline{a}/\emptyset)$.

Corollary

If $C = \mathrm{Ecl}(C)$, then $\delta(\overline{x}/C) \geq \mathrm{Edim}(\overline{x}/C)$.

Schanuel's Conjecture asserts that $\delta(\overline{x}) \geq 0$ for all $\overline{x}$.

Definition

We say that $\overline{a}$ is an *essential counterexample* if $\delta(\overline{a}) < 0$ and for all $\overline{b} \in \mathrm{span}_{\mathbb{Q}}(\overline{a})$, $\delta(\overline{b}) \geq \delta(\overline{a})$.

If $\overline{a}$ is a counterexample, then there is $\overline{b} \in \mathrm{span}_{\mathbb{Q}}(\overline{a})$ an essential counterexample.

Exponential Algebraicity of Essential Counterexamples

Theorem (Kirby)

Let $\overline{a} \in \mathbb{C}_{\exp}$ be an essential counterexample to Schanuel's Conjecture, then $\overline{a} \in \mathrm{Ecl}(\emptyset)$.

Suppose $\overline{a}$ is an essential counterexamble and $\overline{a} \notin \mathrm{Ecl}(\emptyset)$.

Let $\overline{b}$ be a basis for $\mathrm{span}_{\mathbb{Q}}(\overline{a})$ over $\mathrm{Ecl}(\emptyset)$.

$td(\overline{a}, E(\overline{a})/\overline{b}, E(\overline{b})) \geq td(\overline{a}, E(\overline{a}))/\mathrm{Ecl}(\emptyset))$ and
$ldim_{\mathbb{Q}}(\overline{a}/\overline{b}) = ldim_{\mathbb{Q}}(\overline{a}/\mathrm{Ecl}(\emptyset))$.

Thus $\delta(\overline{a}/\overline{b}) \geq \delta(\overline{a}/\mathrm{Ecl}(\emptyset))$

and by the Corollary $\delta(\overline{a}/\mathrm{Ecl}(\emptyset)) \geq \mathrm{Edim}(\overline{a}/\mathrm{Ecl}(\emptyset)) \geq 1$.

But then $\delta(\overline{b}) < \delta(\overline{a})$ and $\overline{a}$ is not an essential counterexample.

Problem

If there are counterexamples to Schanuel's Conjecture, there are counterexample in $\mathrm{Ecl}(\emptyset)$.

But.....

What is $\mathrm{Ecl}(\emptyset)$ in $\mathbb{C}_{\mathsf{exp}}$? Is it countable?

Khovanskii Systems

For $k \subseteq K$ an exponential subfield, let $k[X_1, \ldots, X_n]^E$ denote all exponential terms over k

For $f_1, \ldots, f_n \in k[X_1, \ldots, X_n]^E$ let $J(\overline{X})$ be the Jacobian matrix $J(\overline{X}) = \left(\frac{\partial f_i}{\partial X_j}(\overline{X}) \right)$.

Definition

Suppose $A \subset K$ and k is the exponential field generated by A. We say that a_1 is in the *Khovanskii exponential closure* of A if there are $a_2, \ldots, a_n \in K$ and $f_1, \ldots, f_n \in k[X_1, \ldots, X_n]^E$ such that such that

$$f_1(\overline{a}) = \cdots = f_n(\overline{a}) = 0 \text{ and } \det J(\overline{a}) \neq 0.$$

We say $a_1, \ldots, a_n \in \mathrm{ecl}(A)$.

Advantages of ecl

Suppose $F \subset K$ are exponential fields and $A \subset F$.
Then $\mathrm{ecl}^F(A) \subseteq \mathrm{ecl}^K(A)$.

Work in $\mathbb{C}_{\exp}$
Suppose $f_1(\bar{a}) = \cdots = f_n(\bar{a}) = 0$ and $\det J(\bar{a}) \neq 0$.

By the Inverse Function Theorem, there is an open neighborhood U of $\bar{a}$ the function $\bar{x} \to (f_1(\bar{x}), \ldots, f_n(\bar{x}))$ is invertible.

Thus the solutions to the Khovanskii system are isolated.

Corollary

In $\mathbb{C}_{\mathrm{exp}}$
i) If $A \subset \mathbb{C}$ *is countable, then* $\mathrm{ecl}(A)$ *is countable.*
ii) If $a \in \mathrm{ecl}(\emptyset)$, *then* a *is computable.*

What is the relationship between Ecl and ecl?

ecl ⊂ Ecl

Lemma

If $f \in k[X_1, \ldots, X_n]^E$*,* $f(\overline{a}) = 0$ *and* $D \in \mathrm{Der}(K/k)$*, then* $\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\overline{a}) D(a_i) = 0$.

Let k be the exponential field generated by A and let $f_1, \ldots, f_n$ be terms over k such that $f_1(\overline{b}) = \cdots = f_n(\overline{b}) = 0$ and $\det J(\overline{b}) \neq 0$
By the Lemma

$$J(\overline{a}) \begin{pmatrix} D(b_1) \\ \vdots \\ D(b_n) \end{pmatrix} = 0$$

Since $J(\overline{a})$ is invertible, $D(b_1) = \cdots = D(b_n) = 0$ and $\overline{b} \in Ecl(A)$.

Corollary

$\mathrm{ecl}(A) \subseteq \mathrm{Ecl}(A)$.

$\mathrm{ecl} = \mathrm{Ecl}$

Theorem (Kirby)

$\mathrm{ecl}(A) = \mathrm{Ecl}(A)$.

The proof is a careful analysis of extensions of exponential derivations.

Corollary

All essential counterexamples to Schanuel's conjecture are in $\mathrm{ecl}(\emptyset)$*;*

In $\mathbb{C}_{\mathrm{exp}}$ *there are countably many possible essential counterexamples, all of which are recursive.*

Corollary

Schanuel's Conjecture is true if and only if there are no recursive counterexamples.

This can be written as a Π^0_3*-sentence.*

Thank You

References

J. Ax, On Schanuel's conjectures. Ann. of Math. (2) 93 1971, 252–268.

J. Kirby, Exponential Algebraicity in Exponential Fields, Bull. Lond. Math. Soc. 42 (2010), no. 5, 879–890.

Further Digression–Exponential Differential Forms

Let $\Omega^E(K/A)$ be the K-vector space constructed begining with the vector space generated by the formal differential forms dx for $x \in K$ modulo the vector space generated by the relations:

- da, for $a \in A$
- $d(x+y) - dx - dy$
- $d(xy) - xd(y) - yd(x)$
- $d(E(x)) - E(x)dx$.

We have a derivation $d : K \to \Omega^E(K/A)$ taking x to the image of dx.

Lemma (Universal Property)

- *If $\eta : \Omega^E(K/A) \to K$ is K-linear, then $D = \eta \circ d \in EDer(K/A)$.*
- *Moreover, if $D \in EDer(K/A)$, there is a unique η_D such that $D = \eta_D \circ d$.*

Proof of Finite Character

Suppose $b \in \mathrm{Ecl}(A)$. Then db must be 0 in $\Omega^E(K/A)$.

We can find $a_1, \ldots, a_m \in A$ such that db is in the vector space generated by $da_1, \ldots, da_n$ and finite many relations $d(x+y) - dx - dy$, $d(xy) - xd(y) - yd(x)$, $d(E(x)) - E(x)dx$.

Then db is still zero in $\Omega^E(K/a_1, \ldots, a_m)$ and for $D \in \mathrm{EDer}(K/a_1, \ldots, a_n)$, then $D(a) = 0$.

Go Back