Fifty Years in the Model Theory of Differential Fields

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Basics

A *differential field* is a field K of characteristic 0 with a derivation $D : K \to K$

$$D(a+b) = D(a) + D(b) \text{ and } D(ab) = aD(b) + bD(a).$$

A *differential polynomial* in variables $X_1, \ldots, X_n$ over K is an element of the ring $K\{X_1, \ldots, X_n\}$ which is

$$K[X_1, \ldots, X_n, D(X_1), \ldots, D(X_n), \ldots, D^{(m)}(X_1), \ldots, D^{(m)}(X_n), \ldots].$$

The *order* of $f \in K\{X_1, \ldots, X_n\}$ is the largest m such that some $D^{(m)}$ occurs.
The constant subfield of K is $C_K = \{x \in K : D(x) = 0\}$.

Prehistory

A differential field K is *existentially closed* if for any finite system Σ of polynomial differential equations having a solution in some $L \supseteq K$ already has a solution in K.

Theorem (Robinson 1959)

The theory of existentially closed differential fields T is complete, model complete and decidable.

Robinson's axiomatization was quite difficult and built on theory of differential ideals developed by Ritt and Kolchin.

Blum's Thesis 1968

Definition

We say that (K, D) is *differentially closed* $K \models \mathrm{DCF}$ if
• K is an algebraically closed field of characteristic zero;
• If $f(X), g(X) \in K\{X\}$ are nonzero and $\mathrm{order}(f) > \mathrm{order}(g)$, then there is $x \in K$ such that $f(x) = 0 \wedge g(x) \neq 0$.

Theorem (Blum)

DCF is a complete theory with quantifier elimination axiomatizing the theory of existentially closed differential fields.

ω-stability

Let $K \models$ DCF and p be a 1-type over a differential field $k \subseteq K$.
By quantifier elimination p is determined by
$I_p = \{f \in k\{X\} : f(x) = 0 \in p\}$, a prime differential ideal.
$p \mapsto I_p$ is a bijection between $S_1(k)$ and prime differential ideals of $k\{X\}$.

Theorem (Ritt–Raudenbush Basis Theorem)

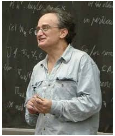
Any radical differential ideal in $k\{X_1, \ldots, X_n\}$ is finitely generated.

Thus $|S_1(k)| = |k|$.

Corollary (Blum)

DCF is ω-stable.

Differential Closures



Definition

We say that $K \supseteq k$ is a *differential closure* of k if K is differentially closed and for any differentially closed $L \supseteq k$ there is a differential embedding $j : K \to L$ fixing k pointwise.

Blum observed differential closures $=$ prime model extensions in DCF.

Theorem

In ω-stable theories:
[Morley 1965] *prime model extensions exist;*
[Shelah 1972] *prime model extensions are unique; i..e., if $\mathcal{M}$ and $\mathcal{N}$ are prime over A there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ fixing A.*

Differential Closures



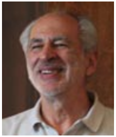
Corollary

Any differential field k has a differential closure K and any two differential closures of k are isomorphic over k.

In some ways differential closures behave like algebraic closures or real closures but in other ways they do not. Differential closures need not be minimal.

Theorem (Kolchin/Rosenlicht/Shelah 1974)

Differential closures need not be minimal. For example, if K is a differential closure of $\mathbb{Q}$ then there is $L \subset K$ with L isomorphic to K.



The Least Misleading Example

Gerald Sacks in his 1972 book *Saturated Model Theory* described differentially closed fields as the *least misleading example* of an ω-stable theory.

Reasons to Study DCF

- Many interesting phenomena from pure model theory, particularly geometric stability theory, have found natural manifestations in differentially closed fields.
- Model theoretic methods have provided useful insights into differential algebraic geometry and differential Galois theory.
- Model theoretic and differential algebraic methods have combined in applications to number theory (diophantine and transcendence questions)

Differential Algebraic Groups

If K is a differential field we can define the *Kolchin topology* on K^n, the smallest topology where $\{x \in K^n : p(x) = 0\}$ are closed for $p \in K\{X_1, \ldots, X_n\}$.
The group objects in this topology are the *differential algebraic groups*.

There is a good general theory of stable/ω-stable groups that can be used to analyze differential algebraic groups

Theorem (Pillay 1997)

If G is a differential algebraic group, then there is a differential algebraic group embedding of G into an algebraic group H.

Differential Galois Theory

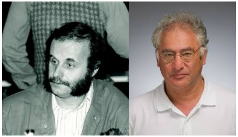


Let $L(X) = a_n D^{(n)}(X) + \cdots + a_1 D(X) + a_0 X$.
Let k be a differential field with C_k algebraically closed and let K/k be a differential closure. Then $C_K = C_k$.
The solutions to $L(X) = 0$ in K from an n-dimensional vector space over C_K . Let $f_1, \ldots, f_n$ be a basis and let $l = k\langle f_1, \ldots, f_n \rangle$. We call l a *Picard–Vessiot* extension of k.
Let $\mathrm{Gal}_D(l/k)$ be the differential automorphisms of l fixing k pointwise.

Theorem (Lie/Kolchin)

$\mathrm{Gal}_D(l/k)$ is isomorphic to the C_k points of a linear algebraic group defined over C_k.

Differential Galois Theory

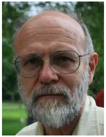


Kolchin generalized this to the class of *strongly normal* extensions for which the differential Galois groups are exact the general algebraic groups over C_k.

Poizat (1983) gave a model theoretic proof of Kolchin's result by showing that the group $\mathrm{Gal}_D(l/k)$ is isomorphic to a definable group in K which is a definable subset of C_K^m.
This paper also introduced the model theoretic ideas of *elimination of imaginaries*.

Pillay (1997-98) generalized Kolchin's strongly normal theory and found a natural class of extensions where $\mathrm{Gal}_D(l/k)$ can be an arbitrary finite dimensional differential algebraic group.

Strongly Minimal Sets



Definition

A definable set $X \subseteq \mathcal{M}^n$ is *strongly minimal* if X is infinite and for every definable $Y \subset X$ either Y or $X \setminus Y$ is finite.

General Examples:

- ACF: K an algebraically closed field and $X \subseteq K^n$ an irreducible algebraic curve ($\pm$ finitely many points).
- DCF: $\mathbb{K}$ differentially closed, C the field of constants.
- Equality: $\mathcal{M}$ an infinite set with no structure and $X = \mathcal{M}$.
- Successor: $\mathcal{M}$ an infinite set $f : \mathcal{M} \to \mathcal{M}$ a bijection with no finite orbits, $X = \mathcal{M}$.
- DAG: $\mathcal{M}$ a torsion free divisible abelian group, $X \subseteq \mathcal{M}^n$ a translate of a one-dimensional subspace defined over $\mathbb{Q}$.

Model Theoretic Algebraic Closure

Definition

If $a \in \mathcal{M}$, $B \subset \mathcal{M}$, a is *algebraic* over B if there is a formula $\phi(x, y_1, \ldots, y_m)$ and $\mathrm{b} \in B^m$ such that $\phi(a, \mathrm{b})$ and $\{x \in \mathcal{M} : \phi(x, \mathrm{b})\}$ is finite.
Let $\mathrm{cl}(B) = \{a : a \text{ algebraic over } B\}$.

- ACF: $\mathrm{cl}(A)=$ algebraic closure of field generated by A.
- DCF: $\mathrm{cl}(A)=$algebraic closure of differential field generated by A.
- equality: $\mathrm{cl}(A) = A$.
- Successor: $\mathrm{cl}(A) = \bigcup_{a \in A}$ orbit of a
- DAG: $\mathrm{cl}(A) = \mathrm{span}_{\mathbb{Q}}(A)$.

Geometry of Strongly Minimal Sets

Definition

A strongly minimal set X is *geometrically trivial* or *degenerate* if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$ for all $A \subseteq X$.

Equality and Successor are geometrically trivial

Definition

A strongly minimal set X is *modular* if $c \in \mathrm{cl}(B \cup \{a\})$, then $c \in \mathrm{cl}(b, a)$ for some $b \in cl(B)$, for all $a \in X$, $B \subseteq X$.

DAG is non-trivial modular: If $c = \sum m_i b_i + na$ where $m_i, n \in \mathbb{Q}$, then $c = b + na$ where $b = \sum m_i b_i$.

ACF is non-modular

When are two strongly minimal sets "the same"?

Definition

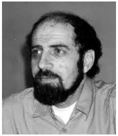
Two strongly minimal sets X and Y are *non-orthogonal* ($X \not\perp Y$) if there is a definable finite-to-finite correspondence $R \subseteq X \times Y$.

Idea: "non-orthogonal"= intimately related, "orthogonal"=not related.

In ACF: If X is a curve there is $\rho : X \to K$ rational so $X \not\perp K$.

In DCF: If X and Y are strongly minimal sets defined over a differentially closed field K, then $X \perp Y$ if and only if for $a \in X(\mathbb{K}) \setminus X(K)$, $Y(K\langle a\rangle^{\mathrm{dcl}}) = Y(K)$–i.e., adding points to X does not force us to add points to Y.

Zilber's Principle



Zilber's Principle: Complexity of the combinatorial geometry is an avatar of algebraic structure.

trivial strongly minimal sets have no infinite definable groups

modular strongly minimal sets are controlled by groups.

Theorem (Hrushovski)

If X is a nontrivial modular strongly minimal set, there is an interpretable modular strongly minimal group G such that $X \not\perp G$.

Zilber Conjectured that non-modular strongly minimal sets only occur in the presence of an algebraically closed field, but Hrushovski refuted this in general.

Strongly Minimal Sets in DCF–Early Results



- The field of constants C is non-locally modular. Indeed if $X \subseteq C^n$ is definable in K, X is definable in C.
- There are many trivial strongly minimal sets

Theorem (Kolchin/Rosenlicht/Shelah 1974)

The differential equation $y' = y^3 - y^2$. Defines a trivial strongly minimal set. If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are distinct solutions with $a_i, b_i \neq 0, 1$, then there is an automorphism σ of $\mathbb{K}$ with $\sigma(a_i) = b_i$.

Zilber's Principle for DCF

Theorem (Hrushovski-Sokolović)

If $X \subseteq \mathbb{K}^n$ is strongly minimal and non-locally modular, then $X \not\perp C$.

The original proof used the high powered model theoretic machinery of *Zariski Geometries* developed by Hrushovski and Zilber.

This was later given a more elementary conceptual proof by Pillay and Ziegler.

Nontrivial Modular Strongly Minimal Sets in DCF

Where do we look for nontrivial modular strongly minimal sets?

- By Hrushovski's result we should look for a modular strongly minimal group G.
- By Pillay's result we may assume that $G \subseteq H$ where H is an algebraic group.
- By strong minimality we may assume that H is commutative and has no proper infinite algebraic subgroups.
- If H is $\mathbb{G}_a$, G must be a finite dimensional C-vector space so $G \not\perp C$.
- If H is $\mathbb{G}_m$ or a simple abelian variety defined over C, either
  - $G \subseteq H(C)$ and $G \not\perp C$, or
  - $G \cap H(C)$ is finite. In this case let $l : H \to \mathbb{K}^d$ be the logarithmic derivative. Then $G \not\perp l(G)$ is a a finite dimensional C vector space and $G \not\perp C$.

Modular Strongly Minimal Sets in DCF

Theorem (Hrushovski-Sokolović)

- *If A is a simple abelian variety that is not isomorphic to one defined over C and $A^\sharp$ is the Kolchin-closure of the torsion points, then $A^\sharp$ is a modular strongly minimal set.*
- *If X is a modular strongly minimal set, then there is A as above such that $X \not\perp A^\sharp$.*
- *$A_0^\sharp \not\perp A_1^\sharp$ if and only if A_0 and A_1 are isogenous.*

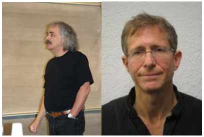
The key tool is the Buium–Manin homomorphism, a differential algebraic homomorphism $\mu : A \to \mathbb{K}^n$ such that $\ker(\mu) = A^\sharp$ and the result that $A^\sharp$ is Zariski dense and has no proper proper infinite differential algebraic subgroups.

Application: The number of countable DCF

In 1984 Shelah proved Vaught's Conjecture for ω-stable theories, but it took almost ten years to show $I(\mathrm{DCF}, \aleph_0) = 2^{\aleph_0}$.

We can assign a dimension to $A^\sharp$ which can be finite or infinite. As these dimensions can be assigned independently we can code graphs into DCF. (eni-DOP)

Diophantine Applications



The strongly minimal sets $A^\sharp$ play a fundamental role in Buium's and Hrushovski's proofs of the Mordell-Lang Conjecture for function fields in characteristic 0.

Corollary

If A is a simple abelian variety not isomorphic to a variety defined over C with $\dim(A) \geq 2$ and $X \subset A$ is a curve, then X contains only finitely many torsion points.

Proof Since $X \cap A^\sharp$ is infinite and $A^\sharp$ is strongly minimal, $X \cap A^\sharp$ is cofinite in $A^\sharp$ and hence Zariski dense in A, a contradiction.

Trivial Pursuits


So far there is no good theory of the trivial strongly minimal sets.

Look for examples:

- Rosenlicht style examples: $y' = f(y)$, f a rational function over C. We can determine triviality by studying the partial fraction decomposition of $1/f$.
- Hrushovski–Itai 2004: For X a curve of genus at least 2 defined over C there is a trivial $Y \subset X$ such that $\mathbb{K}(X) = \mathbb{K}\langle Y \rangle$. They use these ideas to build many different superstable theories of differential fields.



Generic Painlevé Equations

Painlevé began the classification of second order differential equations where the only movable singularities are poles. The classification gives rise six families:

For example $P_{II}(\alpha) : D^{(2)}Y = 2Y^3 + tY + \alpha$ where $D(t) = 1$.

Many questions arise about algebraic relations between solutions to an individual equation and relations between solutions of different equations.

These were attacked by Nishioka, Umemura, Wantanabe...

Nagloo and Pillay 2014 gave a model theoretic interpretation of this earlier work and used model theoretic ideas to significantly extend it. For example,

Theorem

If $\alpha \in \mathbb{C}$ is transcendental, then the solution set of $P_{II}(\alpha)$ is strongly minimal trivial and if $y_1, \ldots, y_n$ are distinct solutions, $y_1, \ldots, y_n$ are algebraically independent.

$\aleph_0$-categoricity

Is there any structure theory for trivial strongly minimal sets in DCF

It was conjectured that if X is a trivial strongly minimal set and $A \subset X$ is finite, then $\mathrm{cl}(A)$ is finite.

Freitag and Moosa 2017, extending earlier unpublished work of Hrushovski, showed that this is true for order 1 strongly minimal sets. One tool of the proof is of independent interest–here is a simple case.

Theorem (Hrushovski)

Suppose V is an irreducible Kolchin closed set of order 2 defined over C such that there are infinitely many irreducible Kolchin closed $X \subset V$ of order 1 defined over C. Then there is a nontrivial differential rational $f : V \to C$, in which case $\{f^{-1}(c) : c \in C\}$ is a family of Kolchin closed subsets.

The j-function

Theorem (Freitag–Scanlon)

There is a third order non-linear differential equation $E(y)$ satisfied by the j-function is strongly minimal, trivial but not $\aleph_0$–categorical.

The proof relies on Pila's Ax–Lindemann–Wierstrass Theorem for the j-function with derivates.

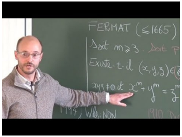
Theorem (Pila)

Let $x_1, \ldots, x_n$ be functions from a variety into the upper half plane satisfying no modular relations. Then

$$j(x_1), \ldots, j(x_n), j'(x_1), \ldots, j'(x_n), j''(x_1), \ldots, j''(x_n)$$

are algebraically independent.

Fushsian Groups




Let Γ be a Fuchsian group of genus 0. Consider a uniformizing function j_Γ such that for the action of Γ on $\mathbb{H}$.

Theorem (Casale, Freitag, Nagloo 20??)

Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety defined over $\mathbb{C}$ and let $t_1, \ldots, t_n \in \mathbb{C}(V)$ taking values in $\mathbb{H}$ at p such that there is no relation $t_i = \gamma t_j$ for $i \neq j, \gamma \in \Gamma$. Then

$$j(x_1), \ldots, j_\Gamma(x_n), j'_\Gamma(x_1), \ldots, j'_\Gamma(x_n), j''_\Gamma(x_1), \ldots, j''_\Gamma(x_n)$$

are algebraically independent over $\mathbb{C}(V)$.

The function j_Γ satisfies a trivial strongly minimal order 3 differential equation.

Thank You