Introductory Lectures
Model Theory III: Tameness

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Two notions of tameness we've seen

- T is *strongly minimal* if in every $\mathcal{M} \models T$ every definable subset of $\mathcal{M}$ is either finite or cofinite.
- If $\mathcal{L} = \{<, \ldots\}$, then T is *o-minimal* if in every $\mathcal{M} \models T$ every definable subset of $\mathcal{M}$ is a finite boolean combination of points and intervals with endpoints in $\mathcal{M} \cup \{\pm\infty\}$.

In this lecture we will introduce several other notions of tameness and interplay between tameness and field theory

o-minimal ordered fields

Real closed fields are o-minimal.

Theorem (Pillay–Steinhorn)

An o-minimal ordered field is real closed.

Let $(F, <)$ be an ordered field. Suppose $f(X) \in K[X]$, $a < b$, $f(a) < 0$ and $f(b) > 0$.
Consider $X = \{x \in (a, b) : f(x) < 0\}$. Since X is open, $b \notin X$ there is $a < c < b$: $c \notin X$ and $(a, c) \subseteq X$.
We must have $f(c) = 0$.

Types

Let $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$. Let $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{b} = (b_1, \ldots, b_n) \in \mathcal{N}$.

$\mathrm{tp}(\mathrm{b}/A) = \{\phi(v_1, \ldots, v_n) : \mathcal{N} \models \phi(\mathrm{b}), \phi$ with parameters from $A\}$.

Let $S_n(A)$ be the set of all types in n variables over A.

Let $K \models \mathrm{ACF}$, $k \subset K \subset L$, $L \models \mathrm{ACF}$ and $\mathrm{b} \in L^n$.

By quantifier elimination $p = \mathrm{tp}(\mathrm{b}/k)$ is determined by

$$I_p = \{f \in k[\mathrm{X}] : \text{“}f(\mathrm{z}) = 0\text{”} \in p\}$$

a prime ideal of $k[\mathrm{X}]$.

Moreover for any prime ideal $J \subset k[\mathrm{X}]$, there is $p \in S_n(k)$ with $J = I_p$

Let p be the type of $X_1/J, \ldots, X_n/J$ in $k[\mathrm{X}]/J^{\mathrm{alg}}$.

κ-stability

We say T is κ-stable if $|S_n(A)| = \kappa$, whenever $|A| = \kappa$.

Fact: If T is $\aleph_0$-stable, then T is κ-stable for all infinite κ.
(Note: We usually say ω-stable instead of $\aleph_0$-stable)

By the Hilbert Basis Theorem, if k is a field every ideal in in $k[X_1, \ldots, X_n]$ is finitely generated.

If k is infinite, the number of prime ideals in $k[X_1, \ldots, X_n]$ is exactly k. Thus $|S_n(k)| = |k|$.

Thus ACF is κ-stable for all infinite κ.

The same is true of any strongly minimal theory.

Our first goal is to prove the converse that every infinite ω-stable field is algebraically closed.

Morley Rank

Let X be a definable set. For α an ordinal

- $\mathrm{RM}(X) \geq 0$ if and only if $X \neq \emptyset$;
- $\mathrm{RM}(X) \geq \alpha + 1$ if and only if there are disjoint definable set $Y_0, Y_1, \ldots$ with $Y_i \subset X$ and $\mathrm{RM}(Y_i) \geq \alpha$;
- For α a limit ordinal $\mathrm{RM}(X) \geq \alpha$ if and only if $\mathrm{RM}(X) \geq \beta$ for all $\beta < \alpha$.

We say $\mathrm{RM}(X) = \infty$ if $\mathrm{RM}(X) \geq \alpha$ for all α.

We say $\mathrm{RM}(X) = \alpha$ if $\mathrm{RM}(X) \geq \alpha$ but $\mathrm{RM}(X) \not\geq \alpha + 1$.

If $\mathrm{RM}(X) = \alpha$ we define the *Morley degree* of X to be the maximal d such that there are disjoint definable $Y_1, \ldots, Y_d \subset X$ with $\mathrm{RM}(Y_i) = \alpha$.

Morley Rank and ω-stability

Theorem

T is ω-stable if and only if every definable set has Morley rank $< \infty$.

In ACF, $\mathrm{RM}(X)$ is the dimension of the Zariski closure of X.
Morley rank can be though of a a general (ordinal valued) notion of dimension.

We can extend Morley rank to types.
$\mathrm{RM}(\mathrm{tp}(b/A)) = \min(\mathrm{RM}(X) : b \in X$ and X is definable with parameters from $A)$.

Two properties of Morley rank that we will use:
• $\mathrm{RM}(X \cup Y) = \max(\mathrm{RM}(X), \mathrm{RM}(Y))$.
• If $f : X \to Y$ is definable surjective and finite-to-one, then $\mathrm{RM}(X) = \mathrm{RM}(Y)$.

ω-stable groups–DCC

Suppose $(G, \cdot, \dots)$ is an ω-stable group and $H \subset G$ is a definable subgroup. Then $G = \bigcup_{g \in G} gH$ and $\mathrm{RM}(H) = \mathrm{RM}(gH)$.
If $[G : H] < \aleph_0$, $\mathrm{RM}(H) = \mathrm{RM}(G)$, $\deg(H) < \deg(G)$.
If $[G : H] \geq \aleph_0$, $\mathrm{RM}(H) < \mathrm{RM}(G)$.

Theorem (Baldwin–Saxl)

In an ω-stable group, there is no infinite proper descending chain of definable subgroups.

We say G is *connected* if there are no definable subgroups of finite index.

Corollary

If G is an ω-stable group, there is a definable connected $G^0 \subseteq G$ with $[G : G^0] < \infty$.
G^0 is fixed by every definable group automorphism of G.

ω-stable groups–generic types

Let X be defined by $\phi(\mathrm{x},\mathrm{a})$ with $\mathrm{RM}(X)=\alpha$. There are $\deg(X)$ many p types of rank α with $\phi(\mathrm{x},\mathrm{a})\in p$.
We call such p generic types of X.

If G is an ω-stable group and $H\subset G$ is a definable subgroup with $[G:H]=n$ then $\deg(G)\geq n$ so G has multiple generic types.

Theorem

An ω-stable group G is connected if and only if there is a unique generic type.

ω-stable fields

Let $(K, +, \cdot, \ldots)$ be an infinite ω-stable field.

claim 1 The additive group $(K, +, \ldots)$ is connected.

Suppose not. Let K^0 be the connected component of K.

For $a \in K$, automorphism $x \mapsto ax$ fixes K^0 setwise–but then K^0 is a nontrivial ideal.

claim 2 The multiplicative group $(K^\times, \cdot, \ldots)$ is connected.

By claim 1 there is a unique type of maximal rank in K. This must be the unique generic type for $(K, \cdot)$ as well. Thus $K^\times$ is connnected.

ω-stable fields

claim 3 Every $a \in K$ has an n^{th}-root for all n.

Consider the multiplicative homomorphism $x \mapsto x^n$.
This map is finite to one, so the image K^n has the same rank as $K^\times$.
Since $K^\times$ is connected $K^\times = K^n$.

claim 4 If K has characteristic $p > 0$, then the Artin–Schreier map $x \mapsto x^p + x$ is surjective.

This map is a finite-to-one additive homomorphism.
As above in case 3 it must be surjective.

ω-stable fields

claim 5 Suppose K contains all m^{th} roots of unit for all $m \leq n$. Then K has no Galois extensions of degree n.

Suppose $[L : K]$ is Galois of degree n and $p|n$ is prime.
There is $K \subseteq F \subset L$ such that L/F is a Galois extension of degree p. Let $L = F(\alpha)$.

The field F is interpretable in K and hence it is also ω-stable.
So F is ω-stable and contains all m^{th} roots of unity for $m \leq p$

If $p \neq \text{char}(K)$, the minimal polynomial of α is $X^p - a$. But $x \mapsto x^p$ is surjective, so $X^p - a$ is not irreducible.

If $p = \text{char}(K)$, the minimal polynomial of α is $X^p + X - a$. But $x \mapsto x^p + x$ is surjective, so $X^p + X - a$ is not irreducible.

ω-stable fields

claim 6 K contains all roots of unity.

Suppose K contains all m^{th}-roots of unity for $m < n$ and η is an n^{th} root of unity.

$K(\eta)/K$ is Galois of degree at most $n-1$, so by claim 5 $\eta \in K$.

Theorem (Macintyre)

Every infinite ω-stable field is algebraically closed.

By 5) and 6) an ω-stable field can have no proper algebraic extensions.

Local Notions of Tameness–Stability

A formula $\phi(x, y)$ has the *order property* if for all n there are $a_1, \ldots, a_n$ $b_1, \ldots, b_n$ such that

$$\phi(a_i, b_j)$$

if and only if $i < j$.

For example in any linear order we can ϕ to be the formula $x < y$, $a_1 < a_2 < \cdots < a_n$ and $b_i = a_i$.

Stability

Theorem (Shelah)

Let T be a theory. The following are equivalent

1. *No formula has the order property.*
2. *T is κ-stable for some infinite κ.*
3. *T is κ-stable exactly if $\kappa^{\aleph_0} = \kappa$.*

If any of these equivalent conditions hold, we say T is stable.

Separably closed fields

Let K be a field of characteristic $p > 0$ that is separably closed but not algebraically closed.
Let $e = [K : K^p]$, $e = 2, 3, \dots, \infty$.

Theorem (Ersóv/Wood)

For a fixed e the theory $SCF_{p,e}$ of separably closed fields of characteristic p is a complete stable theory.

Stable Field Conjecture Every infinite stable field is separably closed.

Open Question: Is $\mathbb{C}(t)$ stable?

Recent Progress on the stable field conjecture

Definition

A field K is *large* if for any curve C defined over K if there is a smooth K point, then there are infinitely many K points.

Large fields: Separably closed fields, real closed fields, fields with a non-trivial henselian valuation, pseudofinite fields, pseudo-algebraically closed fields

Non-large fields: number fields, function fields

Theorem (Johnson, Trann, Walsberg, Yi)

Any large stable field is separably closed.

Independence Property

We say that a formula $\phi(\mathrm{x}, \mathrm{y})$ has the *independence property* if for all n there are $\mathrm{a}_1, \ldots, \mathrm{a}_n$ and $(\mathrm{b}_J : J \subseteq \{1, \ldots, n\})$ such that

$$\phi(\mathrm{a}_i, \mathrm{b}_J) \Leftrightarrow i \in J.$$

The independence property is related to the combinatorial property of having infinite Vapnick–Chervonenkis dimension.

Examples:

- edge relation in a random graph
- $|$ in arithmetic where $a_1, \ldots, a_n$ are distinct primes
- (Duret) $\exists z\ x + y = z^2$ in a pseudofinite field

NIP Theories

A theory is *NIP* if no formula has the independence property.

Examples:

- stable theories;
- o-minimal theories;
- Pressburger arithmetic;
- algebraically closed valued fields;
- $\mathbb{Q}_p$;
- henselian valued fields where the residue field has characteristic zero and NIP;
- any theory of colored linear orders

See Gabe Conant's map of the universe at

`www.forkinganddividing.com`

Conjectures on NIP Fields

Shelah Conjecture If K is an infinite NIP field, then either

- K is algebraically closed;
- K is real closed;
- K admits a non-trivial henselian valuation

Henselianity Conjecture If (K, v) is an NIP valued field, then (K, v) is henselian.

Recent Progress

Theorem (Halevi, Hassson, Jahnke)

1. *Shelah's Conjecture* $\Rightarrow$ *Henselianity Conjecture.*
2. *Shelah's Conjecture* $\Rightarrow$ *every infinite NIP field is either separably closed, real closed or admits a non-trivial* definable *henselian valuation*

Theorem (Johnson)

The Henselianity Conjecture is true for NIP valued fields of positive characteristic.

ict-patterns

An *ict pattern* of depth κ is an array

$$(\phi_\alpha(\mathsf{x}, \mathsf{a}_{\alpha,i}) : \alpha < \kappa, i = 0, 1, \ldots)$$

such that for any function $f : \kappa \to \mathbb{N}$

$$\{\phi_\alpha(\mathsf{x}, \mathsf{a}_{f(\alpha)}) : \alpha < \kappa\} \cup \{\neg\phi_\alpha(\mathsf{x}, \mathsf{a}_i) : \alpha < \kappa, i \neq f(\alpha)\}$$

is consistent.

Example: E_0, E_1 independent equivalence relations with infinitely many classes
$a_{0,0}, a_{0,1}, \ldots$ E_0-inequivalent
$a_{1,0}, a_{1,1}, \ldots$ E_1-inequivalent
Then $(E_i(x, a_{i,j}) : i = 0, 1, j = 0, 1, 2, \ldots)$ is an ict-pattern of depth 2.

ict-patterns and dp-finite theories

Theorem (Shelah)

T has the independence property if and only if there are ict-patterns of arbitrarily large depth.

Definition

- T is *dp-minimal* if every ict-patern has depth 1.
- T is *dp-finite* if for some N every ict-patern has depth at most N.

Examples of dp-minimal: o-minimal, strongly minimal, Pressburger arithmetic, $\mathbb{Q}_p$ or finite extensions, algebraically closed valued fields, $K[[t]]$ where K is algebraically closed or real closed

Examples of dp-finite: Hahn field $\mathbb{R}(((\Gamma)))$ where Γ is dp-finite.

Separably closed fields have an ict-pattern of depth $\aleph_0$

Shelah Conjecture for dp-finite fields

Theorem (Johnson)

If K is an infinite dp-finite field, then one of the following holds:

- *K is algebraically closed;*
- *K is real closed;*
- *K admits a definable henselian valuation.*