Decidability of the Natural Numbers with the Almost-All Quantifier

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Abstract

We consider the fragment $\mathcal{F}$ of first order arithmetic in which quantification is restricted to "for all but finitely many." We show that the integers form an $\mathcal{F}$-elementary substructure of the real numbers. Consequently, the $\mathcal{F}$-theory of arithmetic is decidable.

1 Introduction

In this note, we consider the fragment of first order arithmetic in which quantification is restricted to "for all but finitely many" and its negation "there exist at most finitely many." These quantifiers are quite natural, since mathematics is rich with deep theorems asserting that the various potentially infinite sets are, in fact, finite. However, as we describe below, the formal quantifier "for all but finitely many x" is surprisingly weak. In fact, the fragment of arithmetic that it generates is unable to distinguish between the natural numbers and the non-negative real numbers. Consequently, we can use the quantifier elimination and decidability of the theory of real closed fields, to deduce the same facts for this fragment of the theory of arithmetic.

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The proof that we give below is a direct application of the machinery of o-minimality. Even so, the question was originally motivated by recursion theoretic investigations. Typically, first order structures are presented either by specifying their atomic diagrams or by specifying their generators and relations. The former is a recursive (Δ^0_1) presentation of the structure and the latter is a recursively enumerable (Σ^0_1) presentation. The next level in the arithmetic hierarchy is a Π^0_1 presentation of a first order structure. In a Π^0_1 presentation, the atomic truths are those which are never canceled. In particular, if one is attempting to give a nonrecursive Π^0_1 presentation of the natural numbers, one is faced with the problem of canceling an element a's role as a particular number and reassigning a to an arbitrarily large value. Asking about the atomic types of arbitrarily large numbers naturally leads to asking about the almost-all theory of arithmetic.

No matter how the question happened to be asked, the model theory of o-minimal structures provides a direct route to the answer.

2 The Almost-All Theory of Arithmetic

In the natural numbers the quantifier "for all but finitely many" is equivalent to the quantifier "for all sufficiently large". Since we will also consider non-discrete orderings it will be useful to work with the later quantifier.

Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ be the language of ordered rings. Let Q be a new quantifier symbol. If $\mathcal{M} = (M, <, \dots)$ is a linearly ordered structure, we say that

$$\mathcal{M} \models Qx\ \phi \text{ if and only if } \mathcal{M} \models \exists z \forall x > z\ \phi.$$

Let $\mathcal{F}$ be the smallest collection of $\mathcal{L}$-formulas containing all quantifier free formulas and closed under propositional connectives and Q. We view $\mathcal{F}$ as a fragment of first order logic.

Theorem 2.1 *The natural numbers is an $\mathcal{F}$-elementary substructure of the real ordered field.*

Proof We will prove, by induction on complexity of $\mathcal{F}$-formulas, that if $\phi(x_1, \dots, x_m)$ is an $\mathcal{F}$-formula with free variables $x_1, \dots, x_m$ and $n_1, \dots, n_m \in \mathbb{N}$, then

$$\mathbb{N} \models \phi(n_1, \dots, n_m) \text{ if and only if } \mathbb{R} \models \phi(n_1, \dots, n_m).$$

This is clear for atomic formulas and the induction is trivial for Boolean combinations.

Suppose the claim is true for $\phi(x, \bar{y})$. If $\mathbb{R} \models Qx\ \phi(x, \bar{n})$, then there is $r \in \mathbb{R}$ such that $\mathbb{R} \models \phi(x, \bar{n})$ for all $x > r$. If $s \in \mathbb{N}$ and $s \geq r$, then $\mathbb{N} \models \forall x > s\ \phi(x, \bar{n})$.

On the other hand if $\mathbb{N} \models Qx\ \phi(x, \bar{n})$, then there is $r \in \mathbb{N}$ such that for all $s \in \mathbb{N}$ if $s > r$, then $\mathbb{N} \models \phi(s, \bar{n})$ and $\mathbb{R} \models \phi(s, \bar{n})$. Thus $X = \{x \in \mathbb{R} : \mathbb{R} \models \phi(x, \bar{n})\}$ is unbounded. But $\mathbb{R}$ is o-minimal and, hence, X is a finite union of points and intervals. Thus there is $r' \in \mathbb{R}$ such that $(r, +\infty) \subseteq X$ and $\mathbb{R} \models Qx\ \phi(x, \bar{n})$.

In particular $\mathbb{R}$ and $\mathbb{N}$ have the same $\mathcal{F}$-theory. We can use the quantifier elimination and decidability of the theory of real closed fields, to deduce the same facts for the $\mathcal{F}$-theory of $\mathbb{N}$.

Corollary 2.2 *i) The $\mathcal{F}$-theory of the natural numbers is decidable.*
ii) Every $\mathcal{F}$-formula is equivalent in $\mathbb{N}$ to a quantifier free formula.

The proof of Theorem 2.1 can be applied in more general settings. Suppose $\mathcal{R} = (\mathbb{R}, +, \cdot, <, \ldots)$ is an o-minimal expansion of the real field in a language $\mathcal{L}$ and that the natural numbers are an $\mathcal{L}$-substructure of $\mathcal{R}$. Then $\mathbb{N}$ is an $\mathcal{F}$-elementary submodel of $\mathcal{R}$. For example, let $\mathcal{R} = (\mathbb{R}, +, \cdot, <, e, 0, 1)$ where e is the binary function

$$e(x, y) = \begin{cases} x^y & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

The structure $\mathcal{R}$ is a reduct of $\mathbb{R}_{\exp}$ the real field with exponentiation and Wilkie [5] proved that $\mathbb{R}_{\exp}$ is o-minimal. Thus the natural numbers is an $\mathcal{F}$-elementary submodel. Macintyre and Wilkie [3] proved that if Schanuel's Conjecture is true, then the theory of $\mathbb{R}_{\exp}$ is decidable.[1] Thus if Schanuel's Conjecture holds, then the $\mathcal{F}$-theory of $(\mathbb{N}, +, \cdot, <, x^y, 0, 1)$ is decidable.

3 Completeness of the $\mathcal{F}$-Theory of Commutative Ordered Rings.

Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ be the language of ordered rings. We conclude by proving that any two commutative ordered rings are $\mathcal{F}$-elementarily equiva-

[1] Schanuel's Conjecture asserts that if $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, then the field $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, e^{\lambda_1}, \ldots, e^{\lambda_n})$ has transcendence degree at least n over $\mathbb{Q}$.

lent.

We always assume that rings have a multiplicative identity.

Definition 3.1 A commutative ordered ring A is a *real closed ring* if the intermediate value property holds for every polynomial in $A[X]$.

Cherlin and Dickman [2] proved that real closed rings are exactly convex subrings of real closed fields. For example, suppose R is a real closed field containing infinite elements. Then $A = \{x \in R : |x| < n \text{ for some } n \in \mathbb{N}\}$ is a real closed ring that is not a field. Cherlin and Dickman [2] also proved that any two real closed rings that are not fields are elementarily equivalent.

Real closed rings need not be o-minimal. In $(A, +, \cdot, <)$ we can define the monad of infinitesimals as the ideal of noninvertible elements. This is a convex set that is not an interval with endpoints in A.

Definition 3.2 A linearly ordered structure $(M, <, \ldots)$ is *weakly o-minimal* if every subset of M that is definable with parameters from M is a finite union of convex sets.

Dickman [1] noted that all real closed rings are weakly o-minimal. Indeed [4] shows that every weakly o-minimal ring is real closed. Weakly o-minimal structures still satisfy the *Asymptotic Dichotomy Principle*, namely, if $(M, <, \ldots)$ is weakly o-minimal and $X \subseteq M$ is definable, then there is $r \in M$ such that $(r, +\infty) \subseteq X$ or $(r, +\infty) \cap X = \emptyset$. This is enough to adapt the proof of Theorem 2.1 to prove a mild generalization.

Lemma 3.3 *Suppose A is a commutative ordered ring, R is a real closed ring and A is unbounded in R. If $\phi(x_1, \ldots, x_n)$ is an $\mathcal{F}$-formula with free variables $x_1, \ldots, x_n$ and $a_1, \ldots, a_m \in A$, then*

$$A \models \phi(a_1, \ldots, a_m) \text{ if and only if } R \models \phi(a_1, \ldots, a_m).$$

The assumption that A is unbounded in R is essential. Suppose F is a real closed field and R is a proper convex subring. Let $a \in R$ such that $1/a > R$. Then $R \models Qx\ ax < 1$, while this fails in F. While R is not an $\mathcal{F}$-elementary submodel of F, they will be $\mathcal{F}$-elementarily equivalent.

Theorem 3.4 *Any two commutative ordered rings are $\mathcal{F}$-elementarily equivalent.*

Proof Let A be a commutative ordered ring. We will prove that A is $\mathcal{F}$-elementarily equivalent to $\mathbb{Z}$.

Let F be a real closed field such that $A \subset F$ is bounded in F. Let R be the convex hull of A in F and let R_1 be the convex hull of $\mathbb{Z}$. Since R and R_1 are real closed rings that are not fields they are elementarily equivalent.

Let ϕ be any $\mathcal{F}$-sentence, then

$$\begin{aligned} \mathbb{Z} \models \phi \quad &\Leftrightarrow \quad R_1 \models \phi, \text{ by Lemma 3.3} \\ &\Leftrightarrow \quad R \models \phi \\ &\Leftrightarrow \quad A \models \phi \text{ by Lemma 3.3.} \end{aligned}$$

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