§3 Independence

We now give an algebraic definition of independence which will play the model theoretic role of non-forking in ACFA. We will derive enough good properties of independence to deduce that ACFA is simple, indeed supersimple, and independence corresponds to non-forking in simple theories.

Definition We say that A and B are *independent* over C (and write $A \downarrow_C B$) if $\mathrm{acl}_\sigma(A \cup C)$ and $\mathrm{acl}_\sigma(B \cup C)$ are algebraically disjoint over $\mathrm{acl}_\sigma(C)$.

We begin by relating this notion to classical notions of independence from difference algebra. We say a set B is *transformally independent* over A iff $\{\sigma^i(b) : i \in \mathbf{Z}, b \in B\}$ are algebraically independent over $\langle A \rangle_\sigma$. If a is transformally independent over A, we also say a is *transformally transcendental* over A. The *transformal transcendence degree* of B over A is the maximum cardinality of a $B_0 \subseteq B$ which is transformally independent over A. We denote the transformal transcendence degree of B over A by $\mathrm{ttd}(B/A)$.

We let $\deg_\sigma(A/B)$ be the transcendence degree of $\langle A \rangle_\sigma$ over $\langle B \rangle_\sigma$. If a is transformally transcendental over B, then $\deg_\sigma(a/B) = \infty$. Note that if $p(a, \sigma(a), \ldots, \sigma^m(a)) = 0$, then $p^\sigma(\sigma(a), \ldots, \sigma_{m+1}(a)) = 0$, where p^σ is the polynomial obtained by letting σ act on the coefficients of p. Thus $\deg_\sigma(a/B)$ is the largest n such that $a, \sigma(a), \ldots, \sigma^{n-1}(a)$ is algebraically independent over $\langle B \rangle_\sigma$.

Lemma 3.1 Suppose $A \subset B$, $b_1, \ldots, b_m$ are transformally transcendental over B and $\deg_\sigma(\bar{c}/A\bar{b}) = \deg_\sigma(\bar{c}/B\bar{b}) < \infty$. Then $\bar{b}, \bar{c} \downarrow_A B$.

Proof

There is a transcendence base for $\langle A, \bar{b}, \bar{c} \rangle_\sigma$ over $\langle A \rangle_\sigma$ which is still a transcendence base over $\langle B \rangle_\sigma$.

Lemma 3.2 Independece has the following properties.

a) (Symmetry) If $A \downarrow_C B$, then $B \downarrow_C A$.

b) (Finite Character) $A \downarrow_C B$ iff for all finite $B_0 \subseteq B$, $A \downarrow_C B_0$.

c) (Local Character) For all $\bar{a}$ and B there is a finite $C \subseteq B$ such that $\bar{a} \downarrow_C B$.

d) (Monotonicity) If $A \downarrow_C B$ and $B_0 \subseteq B$, then $A \downarrow_{C \cup B_0} B$.

e) (Transitivity) If $D \supseteq B$, then $A \downarrow_C D$ iff $A \downarrow_C B$ and $A \downarrow_B D$.

Proof

Properties a), b), d) and e) hold since the corresponding fact is true for algebraic disjointness in algebraically closed fields.

For c) suppose $\bar{a} = (\bar{b}, \bar{c})$ where $\bar{b}$ is a transformal transcence base for $\bar{a}$ over B. We can find a finite C such that $\deg_\sigma(\bar{c}/C\bar{b}) = \deg_\sigma(\bar{c}/B\bar{b})$. By 3.1 $\bar{a} \downarrow_C B$.

Corollary 3.3 Suppose $A = \{a_\alpha : \alpha < \lambda\}$ and $a_\alpha \downarrow_C \{a_\beta : \beta < \alpha\}$ for all α. Then A is an independent set, ie. $a_\alpha \downarrow_C \{a_\beta : \beta \neq \alpha\}$.

Proof

This is a standard consequence of transitivity.

Lemma 3.4 (existence of free amalgams) Suppose $p(\bar{x})$ and $q(\bar{y})$ are complete types over E. Then there are $\bar{a}$ and $\bar{b}$ realizing p and q respectively such that $\bar{a} \downarrow_E \bar{b}$.

Proof

We may assume $E = \mathrm{acl}_\sigma(E)$ and that if $(b_1, \ldots, b_m)$ realizes $q(\bar{y})$, then $b_i \notin E$. Let $\bar{a}$ realize p. Let $L_1 = \mathrm{acl}_\sigma(\bar{a})$ and $L_2 = \mathrm{acl}_\sigma(\bar{b})$. Working inside a big algebraically closed field we can find F_1 and F_2 such that F_i is isomorphic to L_i and F_1 and F_2 are algebraically disjoint over E. Let σ_i be the automorphism of F_i induced by σ_i. By 1.1 and 1.4 we can find $\tau \in Aut(\mathbf{K})$ such that $(\mathbf{K}, \tau) \models$ ACFA and $\tau | F_i = \sigma_i$. The images of $\bar{a}$ and $\bar{b}$ are independent realizations of p and q in $(\mathbf{K}, \tau)$.

Thus there is a type $r(\bar{x}, \bar{y})$ extending p and q such that realizations of r are independent realizations of p and q. While this type r will not in general be unique, the next result shows that there is a compatibility.

Theorem 3.5 (The Independence Theorem) Suppose $E = \mathrm{acl}_\sigma(E)$, $p_1(x_1)$, $p_2(x_2)$,$p_3(x_3)$, $p_{1,2}(x_1, x_2)$, $p_{1,3}(x_1, x_3)$, and $p_{2,3}(x_2, x_3)$ are types over E such that $p_{i,j}(x_i, x_j)$ implies that x_i and x_j are realizaions of p_i and p_j respectively which are independent over E. There are b_1, b_2, b_3 independent over E such that (b_i, b_j) realizes $p_{i,j}$ for $i < j \leq 3$.

Proof

By 3.4 we can find b_1, a_2, a_3 such that (b_1, a_j) realizes $p_{1,j}$ and a_2 and a_3 are independent over Eb_1. Let $K_1 = \mathrm{acl}_\sigma(Eb_1)$, $K_j = \mathrm{acl}_\sigma(Eb_1 a_j)$ and $F_j = \mathrm{acl}_\sigma(Ea_j)$ for $j = 2, 3$.

Let c_2, c_3 realize $p_{2,3}$. Let $L_j = \mathrm{acl}_\sigma(E, c_j)$, for $j = 2, 3$. Note that $L_2 \downarrow_E L_3$ and L_i is $\mathcal{L}$-isomorphic to F_i. By 3.3, b_1, a_2, a_3 is an independent sequence; so $F_2 \downarrow_E F_3$. This allows us to extend to an $\mathcal{L}$-isomorphisms between $F_2 F_3$ and $L_2 L_3$. While this need not extend to an $\mathcal{L}$-isomorphism of between $\mathrm{acl}(F_2 F_3)$ and $\mathrm{acl}(L_2 L_3)$, we can find $\tau \in Aut(\mathrm{acl}(F_2 F_3))$ such that $\tau | F_2 F_3 = \sigma$ and $(\mathrm{acl}(L_2 L_3), \sigma) \cong (\mathrm{acl}(F_2 F_3), \tau)$.

We will show that we can find an automorphism μ of the composite $K_2 K_3 \mathrm{acl}(F_1 F_2)$ such that $\mu | K_2 K_3 = \sigma$ and $\mu | \mathrm{acl}(F_2 F_3) = \tau$. We can then use 1.1 to extend to (L, μ) a model of ACFA. In this model b_1, a_2, a_3 have the desired properties.

By 1.4 b) it suffices to show that $K_2 K_3$ and $\mathrm{acl}(F_2 F_3)$ are linearly disjoint over $F_2 F_3$.

claim If $c_1, \ldots, c_n$ are algebraic over $F_2 F_3$ and linearly dependent over $K_2 K_3$, then $c_1, \ldots, c_n$ are linearly dependent over $F_2 F_3$.

There are $\bar{d}_2, \bar{d}_3$ with $\bar{d}_j \in K_j$ and rational functions $h_1, \ldots, h_n$ defined over the prime field such that $\sum h_i(\bar{d}_1, \bar{d}_2) c_i = 0$ and not all of the $h_i(\bar{d}_1, \bar{d}_2)$ are zero.

Since $K_j = \mathrm{acl}(K_1, F_j)$. Thus there are $\bar{\alpha} \in K_1$, $\bar{\beta}_j \in K_j$ and $\mathcal{L}^-$ formulas $\psi_j(\bar{x}, \bar{\alpha}, \bar{\beta}_j)$ expressing $\bar{x}$ is algebraic over $\bar{\alpha}, \bar{\beta}_j$, such that $\psi_j(\bar{d}_j, \bar{\alpha}, \bar{\beta}_j)$. Thus

$$\exists \bar{u} \exists \bar{v}\ \psi_2(\bar{u}, \bar{\alpha}, \bar{\beta}_2) \wedge \psi_3(\bar{v}, \bar{\alpha}, \bar{\beta}_3) \wedge \bigvee_{i=1}^{n} h_i(\bar{u}, \bar{v}) \neq 0 \wedge \sum_{i=1}^{n} h_i(\bar{u}, \bar{v}) c_i = 0$$

Since $a_2, a_3 \mathop{\smile\!\!\!\!\!\vert}_E b$, $\mathrm{acl}(F_2, F_3)$ is algebraically disjoint from K_1 over E. Thus $\mathrm{tp}^-(\bar{c}, \bar{\beta}_2, \bar{\beta}_3/K_1)$ is the heir of $\mathrm{tp}^-(\bar{c}, \bar{\beta}_2, \bar{\beta}_3/E)$ and we can find $\bar{a} \in E$ such that

$$\exists \bar{u} \exists \bar{v}\ \psi_2(\bar{u}, \bar{a}, \bar{\beta}_2) \wedge \psi_3(\bar{v}, \bar{a}, \bar{\beta}_3) \wedge \bigvee_{i=1}^{n} h_i(\bar{u}, \bar{v}) \neq 0 \wedge \sum_{i=1}^{n} h_i(\bar{u}, \bar{v}) c_i = 0.$$

The witnesses $\bar{u}$ and $\bar{v}$ are in K_2 and K_3 respectively, so $c_1, \ldots, c_n$ are linearly dependent over $F_2 F_3$.

Corollary 3.5 ACFA is supersimple.

Kim showed that a theory with a notion of independence satisfying 3.2, 3.4 and 3.5 is simple and independence corresponds to non-forking. Since 3.2 c) guarantees non-forking over finite sets, ACFA is supersimple.

§4 Elimination of Imaginaries

As an application of the independence theorem, we will show that ACFA has elimination of imaginaries. We need the following general model theoretic lemma noted by Evans and Hrushovski.

Lemma 4.1 Let M be a large saturated structure. Let X be $\emptyset$-definable, $e \in M$, $E = \mathrm{acl}(e) \cap X$ and $\bar{a} \in X$. There is $\bar{b}$ such that $\mathrm{tp}(\bar{b}/Ee) = \mathrm{tp}(\bar{a}/Ee)$ and $\mathrm{acl}(E\bar{a}) \cap \mathrm{acl}(E\bar{b}) \cap X = E$.

Proof

Let $A = (\mathrm{acl}(Ea) \cap X) \setminus E$. Without loss of generality $\bar{a} \in A$. Let $(a_i : i \in I)$ enumerate A, $\mathbf{x} = (x_i : i \in I)$ and $p(\mathbf{x})$ be the $\mathrm{tp}(A/Ee)$ in the variables $\mathbf{x}$. We will show that

$$p(\mathbf{x}) \cup \{x_i \neq a : i \in I, a \in A\}$$

is consistent. Suppose not. Then there are $i_1, \ldots, i_n \in I$ and $\alpha_1, \ldots, \alpha_n \in A$ such that for any realization $\mathbf{b}$ of $p(\mathbf{x})$ we have $b_{i_1} = \alpha_1 \vee \ldots \vee b_{i_n} = \alpha_n$.

Let G be the group of automorphisms of M fixing $E \cup \{e\}$ pointwise. Let $S_j = \{g \in G : g(a_{i_j}) = \alpha_j\}$. Then S_j is a coset of $H_j = \{g \in G : g(\alpha_j) = \alpha_j\}$ and $\bigcup_{j=1}^{n} S_j = G$. By a lemma of Neumann, no group is a finite union of cosets of infinite index subgroups. Thus $[G : H_j] < \infty$ for some j and α_j is algebraic over Ee. But then $\alpha_j \in E$, a contradiction.

Theorem 4.2 (Elimination of imaginaries) Suppose $K \models \mathrm{ACFA}$ and $e \in K^{\mathrm{eq}}$. There is $\bar{\beta} \in K^n$ such that e and $\bar{\beta}$ are interdefinable.

Proof

There is an $\emptyset$-definable function f and $\bar{a} \in K$ such that $e = f(\bar{a})$. Let $E = \mathrm{acl}^{\mathrm{eq}}(e) \cap K$.

claim 1 There is $\bar{c}$ realizing $\mathrm{tp}(\bar{a}/E)$ with $f(\bar{c}) = e$ and $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{E} \bar{a}$.

Applying lemma 4.3 in K^{eq} we can find $\bar{b}$ such that $\mathrm{tp}^{\mathrm{eq}}(\bar{a}/Ee) = \mathrm{tp}^{\mathrm{eq}}(\bar{b}/Ee)$ and

$$\mathrm{acl}^{\mathrm{eq}}(E\bar{a}) \cap \mathrm{acl}^{\mathrm{eq}}(E\bar{b}) \cap K = E.$$

In particular $\mathrm{acl}_\sigma(E\bar{a}) \cap \mathrm{acl}_\sigma(E\bar{b}) = E$. Choose $\bar{b}$ realizing $\mathrm{tp}(\bar{a}/E)$ such that

i) $\mathrm{acl}_\sigma(E\bar{a}) \cap \mathrm{acl}_\sigma(E\bar{b}) = E$,

ii) $\mathrm{ttd}(\bar{b}/E\bar{a}) = m$ is maximal,

iii) if $\bar{b}_1 \subseteq \bar{b}$ is a transformal transcendence basis (of size m) for $\bar{b}/E\bar{a}$, then $\deg_\sigma(\bar{b}/E\bar{a}\bar{b}_1) = n$ is maximal (by compactness such an n must exist).

By 3.4 we can find $\bar{c}$ realizing $\mathrm{tp}(\bar{b}/E\bar{a})$ such that $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{E\bar{a}} \bar{b}$. We claim that $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{E} \bar{a}$.

Clearly $f(\bar{c}) = e$. Since $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{E,\bar{a}} \bar{b}$,

$$\mathrm{acl}_\sigma(E\bar{b}) \cap \mathrm{acl}_\sigma(E\bar{c}) = \mathrm{acl}_\sigma(E\bar{b}) \cap \mathrm{acl}_\sigma(E\bar{a}) = E.$$

Since $\bar{a}$ and $\bar{b}$ realize the same type over E, the maximality of m ensures that $\mathrm{ttd}(\bar{c}/E\bar{b}) \leq m$. But

$$\mathrm{ttd}(\bar{x}/E\bar{b}) \geq \mathrm{ttd}(\bar{c}/E\bar{a}\bar{b}) = \mathrm{ttd}(\bar{c}/E\bar{a}) = m.$$

So $\mathrm{ttd}(\bar{c}/E\bar{b}) = m$.

Similarly, if $\bar{c}_1$ is the subtuple of $\bar{c}$ corresponding to $\bar{b}_1$ in $\bar{b}$, $\deg_\sigma(\bar{c}/E\bar{b}) = n$. By choice of $\bar{c}$, $\mathrm{ttd}(\bar{c}/E\bar{a}\bar{b}) = m$ and $\deg_\sigma(\bar{c}/E\bar{a}\bar{b}) = n$. Thus $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{E\bar{b}} \bar{a}$.

Suppose $\alpha_1, \ldots, \alpha_n \in \mathrm{acl}_\sigma(E\bar{c})$ are algebraically dependent over $\mathrm{acl}_\sigma(E\bar{a}\bar{b})$. Let V be the locus of $\bar{\alpha}$ over $\mathrm{acl}_\sigma(E\bar{a}\bar{b})$. Since $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{E\bar{a}} \bar{b}$ and $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{E\bar{b}} \bar{a}$, V is defined over $\mathrm{acl}_\sigma(E\bar{a})$ and $\mathrm{acl}_\sigma(E\bar{b})$. Thus the field of definition of V is contained in $\mathrm{acl}_\sigma(E\bar{a}) \cap \mathrm{acl}_\sigma(E\bar{b}) = E$. So V is defined over E and $\bar{\alpha}$ is algebraically dependent over E. [More model theoretically, the canonical basis for $\mathrm{tp}^-(\bar{\alpha}/E\bar{a}\bar{b})$ is contained in $\mathrm{acl}_\sigma(E\bar{a})$ and $\mathrm{acl}_\sigma(E\bar{b})$ and hence in E.] Thus $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{E} \bar{a}\bar{b}$. So $\bar{c}$ is the desired realization of $\mathrm{tp}(\bar{a}/E)$.

claim 2 If $\mathrm{tp}(\bar{b}/E) = \mathrm{tp}(\bar{a}/E)$, then $f(\bar{b}) = e$.

Suppose there is $\bar{b}$ realizing $\mathrm{tp}(\bar{a}/E)$ with $f(\bar{b}) \neq e$. By 3.4 there is $\bar{d}_0$ realizing $\mathrm{tp}(\bar{a}/E)$ with $\bar{d}_0 \mathop{\smile\!\!\!\!\!\!|}_{E} \bar{a}, \bar{b}$. Since $f(\bar{d}_0) \neq f(\bar{a})$ or $f(\bar{d}_0) \neq f(\bar{b})$ and $\bar{a}$ and $\bar{b}$ realize the same type over E, there is a $\bar{d}$ realizing $\mathrm{tp}(\bar{a}/E)$ with $\bar{d} \mathop{\smile\!\!\!\!\!\!|}_{E} \bar{a}$ and $f(\bar{d}) \neq e$.

Let $\bar{c}$ be as in claim 1. Let $p_i = \mathrm{tp}(\bar{a}/E)$, $p_{1,2} = p_{1,3} = \mathrm{tp}(\bar{a}, \bar{c})$ and $p_{2,3} = \mathrm{tp}(\bar{a}, \bar{d})$. These types satisfy the hypothesies of the Independence Theorem. But $p_{1,2} \cup p_{1,3} \cup p_{2,3}$ is inconsistent as it contains the formula $f(x_1) = f(x_2), f(x_1) = f(x_3)$ and $f(x_2) \neq f(x_3)$. This contradiction establishes the claim.

By compactness there is a forumla $\theta(\bar{x}) \in \mathrm{tp}(a/E)$ such that

$$(\theta(\bar{x}) \wedge \theta(\bar{y})) \rightarrow f(\bar{x}) = f(\bar{y}).$$

Thus the formula $\exists \bar{x}\ \theta(\bar{x}) \wedge f(\bar{x}) = z$ shows that $e \in \mathrm{dcl}^{\mathrm{eq}}(E)$. Let $\bar{\alpha} \in E$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar{\alpha})$. Since $\bar{\alpha} \in \mathrm{acl}^{\mathrm{eq}}(e)$, there is a finite set $A = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_m\}$ of conjugates of $\bar{\alpha}$ over e. Then A and e are interdefinable. By elimination of imaginaries in algebraically closed fields, for some n there is $\bar{\beta} \in K^n$ such that A and $\bar{\beta}$ are interdefinable. Then e and $\bar{\beta}$ are interdefinable.

Elimination of imaginaries allows us to show that the fixed field in a pure field. Fix $K \models$ ACFA and $F = \{x \in K : \sigma(x) = x\}$.

Theorem 4.3 Suppose $X \subseteq F^n$ is definable in K. Then X is definable in the structure $(F, +, \cdot)$.

Proof

We first argue that we can define F in K with parameters from F. Let $\phi(\bar{x}, \bar{y})$ be an $\mathcal{L}$ formula such that $X = \{\bar{x} \in K : \phi(\bar{x}, \bar{a})\}$. Let $\sim$ be the equivalence relation $\bar{b} \sim \bar{c}$ iff $\phi(\bar{x}, \bar{b}) \leftrightarrow \phi(\bar{x}, \bar{c})$. Define $e \in K^{\mathrm{eq}}$ by $e = \bar{a}/E$. Then e is a canonical parameter for X and by elimination of imaginaries we find an interdefinable $\bar{e} \in K^m$ for some m. X is definable from $\bar{e}$.

Note that σ is an automorphism of the $\mathcal{L}$ structure (K, σ). Clearly σ fixes the set X setwise (indeed it fixes the elements of X). Thus $\sigma(\bar{e}) = \bar{e}$. So $\bar{e} \in F^m$.

By quantifier elimination X is defined by a formula $\exists \bar{y} \phi(\bar{x}, \bar{y})$ with parameters from F such that whenever $\phi(\bar{a}, \bar{b})$, then $\bar{b} \in \mathrm{acl}_\sigma(\bar{a}) = \mathrm{acl}(\bar{a})$, since $\bar{a} \in F$. By compactness there is a number d such that every every $\phi(\bar{a}, \bar{b})$ implies $\bar{b}$ is contained in a Galois extension of the fixed field of degree d. Since F is pseudofinite there is a unique extension L/F of degree d and

$$\bar{x} \in X \Leftrightarrow (L, \sigma|L) \models \exists \bar{y}\ \phi(\bar{x}, \bar{y}).$$

We can interpret $(L, \sigma|L)$ in the field F. Since $L = F(\alpha)$ for some α we can identify F^d with L via

$$(a_0, \ldots, a_{d-1}) \mapsto \sum_{i=0}^{d-1} a_i \alpha^i.$$

Addition, multiplication and the action of σ on L correspond to operations on F^n definable in $(F, +, \cdot)$. [Note that as σ fixes F, the action of σ is determined by knowing $\sigma(\alpha)$.] Thus there is an $\mathcal{L}^-$ formula $\psi(\bar{x})$ such that

$$F \models \psi(\bar{x}) \Leftrightarrow (L, \sigma|L) \models \exists \bar{y}\ \phi(\bar{x}, \bar{y}).$$

So X is definable in the pure field F.