§5 SU rank and generic types of algebraic varieties

If $p \in S(B)$ and $A \subset B$ we say that p *forks* over A if $a \not\downarrow_A B$ for any realization a of p. We now define the analog of Lascar's U rank for simple theories.

Definition Let $p \in S(B)$. We define the rank SU as follows:

i) $\mathrm{SU}(p) \geq 0$;

ii) $\mathrm{SU}(p) \geq \alpha + 1$ if there is $B \supset A$ and $q \in S(B)$ a forking extension of p with $\mathrm{SU}(q) \geq \alpha$;

iii) $\mathrm{SU}(p) \geq \alpha$ for α a limit ordinal iff $\mathrm{SU}(p) \geq \beta$ for all $\beta < \alpha$.

The following basic lemmas are proved just as they are for U rank.

Lemma 5.1 a) $\mathrm{SU}(p) = 0$ iff p has only finitely many realizations.

b) If $A \subset B$ and $a \downarrow_A B$, then $\mathrm{SU}(a/A) = \mathrm{SU}(a/B)$.

c) $\mathrm{SU}(a/Ab) + \mathrm{SU}(b/A) \leq \mathrm{SU}(ab/A) \leq \mathrm{SU}(a/Ab) \oplus \mathrm{SU}(b/A)$.

If $a \in K$, $A \subset B$ and $\deg_\sigma(a/B) < \deg_\sigma(a/A)$ then $\mathrm{SU}(a/A) > \mathrm{SU}(a/B)$. From this and 5.1a) it follows that for $a \in K$, $\mathrm{SU}(a/A) \leq \deg_\sigma(a/A)$. In particular $\deg_\sigma(a/A) = 1$, then, by 5.1, $\mathrm{SU}(a/A) = 1$ and if $p \in S_1(A)$, then $\mathrm{SU}(p) \leq \omega$ and if a is not transformally transcendental over A, then $\mathrm{SU}(a/A) \leq \deg_\sigma(a/A) < \omega$. We show next that $\mathrm{SU}(p) = \omega$ for p the type of an element transformally transcendetal over A.

Proposition 5.2. a) If a is transformally transcendental over A, then $\mathrm{SU}(a/A) = \omega$.

b) If $a_1, \ldots, a_n$ are transformally independent over A, then $\mathrm{SU}(\bar{a}/A) = \omega n$.

Proof

Define $a_0 = a$ and $a_{n+1} = \sigma(a_n) - a_n$ for $n \geq 1$.

claim Each a_n is transformally transcendental over A and $\deg_\sigma(a_n/Aa_{n+1}) = 1$.

Suppose a_n is transformally transcendental over A. Since

$$\langle A\rangle_\sigma(a_n, \sigma^i(a_{n+1}) : -m \leq i \leq m) = \langle A\rangle_\sigma(\sigma^j(a_n) : -m \leq j \leq m+1),$$

The set $\{a_n\} \cup \{\sigma^i(a_{n+1}) : i \in \mathbf{Z}\}$ is algebraically independent over $\langle A\rangle_\sigma$. Thus a_{n+1} is transformally transcendental and $\deg_\sigma(a_n/Aa_{n+1}) \geq 1$. Since a_n is a solution to $\sigma(x) = x + a_{n+1}$, $\deg_\sigma(a_n/Aa_{n+1}) = 1$.

Since $a_{i+1} \in \mathrm{dcl}(a)$, $\mathrm{SU}(a/A) = \mathrm{SU}(a_0, \ldots, a_n/A)$. We prove by induction that $\mathrm{SU}(a_m, \ldots, a_n/A) \geq n - m + 1$.

By 5.1 c)

$$\mathrm{SU}(a_m, \ldots, a_n/A) \geq \mathrm{SU}(a_m/Aa_{m+1}, \ldots, a_n) + \mathrm{SU}(a_{m+1}, \ldots, a_n/A).$$

Since $\deg_\sigma(a_m/Aa_{m+1}) = 1$, $\mathrm{SU}(a_m/Aa_{m+1}, \ldots, a_n) = \mathrm{SU}(a_m/Aa_{m+1}) = 1$. By induction, $\mathrm{SU}(a_{m+1}, \ldots, a_n/A) \geq n - m$. So $\mathrm{SU}(a/A) = \omega$.

Suppose $E = \mathrm{acl}_\sigma(E)$ and a and b are transformally transcendetal over E. We would like to argue that a and b realize the same 1-type over E. Clearly $\langle Ea\rangle_\sigma \cong \langle Eb\rangle_\sigma$, but we need to know that $\mathrm{acl}_\sigma(Ea) \cong \mathrm{acl}_\sigma(Eb)$ (here we are talking about $\mathcal{L}$-isomorphisms). We will prove this in a bit more generality.

b) follows from a) using 5.1 c).

Lemma 5.3 Suppose $E = \mathrm{acl}_\sigma(E)$ and $a, b \in K^n$ realize the same quantifier free type over E. Then $\mathrm{tp}(a/E) = \mathrm{tp}(b/E)$ iff and only if every automorphism of $\mathrm{acl}_\sigma(Ea)$ which agrees with σ on $\langle Ea\rangle_\sigma$ is conjugate to σ under the action of the $Gal(\mathrm{acl}_\sigma(Ea)/\langle Ea\rangle_\sigma$.

Proof

Clearly $\langle Ea\rangle_\sigma \cong \langle Eb\rangle_\sigma$. Identifying $\langle Ea\rangle_\sigma$ and $\langle Eb\rangle_\sigma$, it suffices to show that if $\tau \in Aut(\mathrm{acl}_\sigma(Ea))$ and $\tau \supset \sigma|\langle Ea\rangle_\sigma$, then there is $f \in Gal(\mathrm{acl}_\sigma(Ea)/\langle Ea\rangle_\sigma)$ such that f is an $\mathcal{L}$-isomorphism between $(\mathrm{acl}(Ea), \sigma)$ and $(\mathrm{acl}(Ea), \tau)$. But then $\sigma = f^{-1}\tau f$.

We use the following Galois theoretic lemma (for the proof see Chatzidakis-Hrushovski).

Lemma 5.4 Let E be a field and $\widetilde{E}$ be the algebraic closure of E. Suppose $\sigma_0 \in Aut(E)$, $\sigma \in Aut(\widetilde{E})$ and $\sigma_0 \subset \sigma$. Then the following are equivalent:

i) If $\tau \in Aut(\widetilde{E})$ and $\sigma_0 \subset \tau$ then τ is conjugate to σ under the action of the Galois group $Gal(\widetilde{E}/E)$.

ii) There is no proper finite Galois extension L/E such that $\sigma(L) = L$.

We now describe the general setting we will be working in for the rest of this section. Suppose $E = \mathrm{acl}_\sigma(E)$ and $V \subseteq K^n$ is an irreducible variety defined over K of dimension d. Without loss of generality we may assume that a generic point of V is of the form $(a_1, \ldots, a_d, \bar{b})$ where $a_1, \ldots, a_d$ are algebraically independent over E and $\bar{b}$ is algebraic over $E(\bar{a})$. Let

$$\Gamma_{V,E}(\bar{x}, \bar{y}) = \mathrm{tp}^-(\bar{a}, \bar{b}) \cup \{p(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^n(\bar{x})) \neq 0 : p \in E[\bar{X}, \bar{X}_1, \ldots, \bar{X}_m] \setminus \{0\}, m \in \mathbf{N}\}.$$

Lemma 5.5 If β and β' realize $\Gamma_{V,E}$, then β and β' realize the same quantifier free type over E.

Proof

Suppose p is a polynomial over E and $p(\beta, \sigma(\beta), \ldots, \sigma^m(\beta)) = 0$. Since $(\beta, \ldots, \sigma^m(\beta))$ has transcendence degree $d(m+1)$ over E it is a generic point of $V \times \sigma(V) \times \cdots \times \sigma^m(V)$. Thus $p(\bar{X}, \sigma(\bar{X}), \ldots, \sigma^n(\bar{X}))$ vanishes at all points of V. It follows that β and β' realize the same quantifier free type over E.

Proposition 5.6 Suppose $(\bar{a}, \bar{b})$ is a realization of Γ_V. Then $\langle E\bar{a}\bar{b}\rangle_\sigma$ has no proper finite Galois extension L with $\sigma(L) = L$.

Proof

Suppose L is a finite Galois extension of $\langle E\bar{a}\bar{b}\rangle_\sigma$. There is $\alpha \in L$ such that $L = \langle E\bar{a}\bar{b}\rangle_\sigma(\alpha)$. Choose i and j such that α and $\bar{b}$ are algebraic over

$$E_1 = E(\sigma^i(\bar{a}), \ldots, \sigma^j(\bar{a})).$$

Let $\tau = \sigma^{j-i+1}$ and let $E_2 = E(\sigma^k(\bar{a}) : k < i \text{ or } k > j)$. Then $\tau(\alpha)$ is algebraic over E_2 but $\tau(\alpha) \notin E_2$.

Since $\bar{a}$ is transformally independent over E, E_1 and E_2 are algebraically disjoint over E and hence $E_1(\alpha, \bar{b})$ and $E_2(\tau(\alpha))$ are algebraically disjoint over E and hence linearly disjoint over E. It follows (by transitivty of linear disjointness) that $L = E_2E_1(\alpha, \bar{b})$ and $E_2(\tau(\alpha))$ are linearly disjoint over E_2. Thus $\tau(\alpha) \notin L$ and $\sigma(L) \neq L$.

Corollary 5.7 Let E and V be as above. a) If α and β realize $\Gamma_{V,E}$, then $\mathrm{tp}(\alpha/E) = \mathrm{tp}(\beta/E)$ and $\mathrm{SU}(\alpha/E) = \omega d$. Let $p_{V,E} = \mathrm{tp}(\alpha/E)$. This is the unique type in V of maximal SU-rank.

b) If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are transformally transcendental over E, then $\mathrm{tp}(\bar{a}/E) = \mathrm{tp}(\bar{b}/E)$ and $\mathrm{SU}(\bar{a}/E) = \omega n$.

c) Suppose $\mathrm{tp}^-(\bar{c}/E) = \mathrm{tp}^-(\bar{d}/E)$ and $\mathrm{RM}(\mathrm{tp}^-(\bar{c}/E)) = \mathrm{ttd}(\bar{d}/E) = \mathrm{ttd}(\bar{c}/E)$. Then $\mathrm{tp}(c/E) = \mathrm{tp}(d/E)$.

d) If $F \supseteq E$ be a difference field, then $p_{V,F}$ is the unique non-forking extension of $p_{V,E}$ to F.

Proof

a) The first two claims follow from 5.3-5.6 and 5.1 c). If $\alpha \in V$ and α does not realize $\Gamma_{V,E}$, then either $\mathrm{tp}^-(\alpha/E)$ is not the generic type of V in which case $\alpha \in W$ for some proper Zariski closed $W \subset V$ and $\mathrm{SU}(\alpha/E) \leq \omega(\dim W) \leq \omega(d-1)$, or $\mathrm{ttd}(\alpha/E) < d$ in which case $\mathrm{SU}(\alpha/E) < \omega d$ by 5.1 c).

b) is a special case of a)

c) is a restatement of a)

d) is clear from a)

We call $p_{V,E}$ the *σ-generic type* of V over E. We next show that σ-generic types are definable.

Proposition 5.8 Let $E = \mathrm{acl}_\sigma(E)$ and let $V \subseteq K^n$ be an irreducible algebraic variety defined over E. Then $p_{V,E}$ is definable over E.

Proof

Let $M \supset E$ be a $|E|^+$-saturated model of ACFA. Let $p = p_{V,M}$. We first show that p is definable over M. This is just the standard argument (ie. from Pillay's book) that uniqueness of non-forking extensions implies definablility. For each formula $\phi(\bar{x}, y_1, \ldots, y_m)$ let

$$P_\phi = \{\bar{b} \in M^m : \phi(\bar{x}, \bar{b}) \in p\}.$$

Let $L' \supset L$ be obtained by adding constant symbols for each element of M and predicate symbols for each P_ϕ. Let T' be the $\mathcal{L}'$-theory of M, let $N' \models T'$ and let N be the $\mathcal{L}$-reduct of N. Then N is an elementary extension of M as

$$N \models \psi(\bar{m}) \Leftrightarrow N \models P_\psi(\bar{m}) \Leftrightarrow M \models P_\psi(\bar{m}) \Leftrightarrow \psi(\bar{m}).$$

Let

$$q(\bar{x}) = \{\phi(\bar{x}, \bar{n}) : N \models P_\psi(\bar{n})\}.$$

It is easy to see that q is a complete type and $q \supset \Gamma_{V,N}$. Thus $q = p_{V,N}$. It follows that N' is the unique expansion of $(N, m)_{m \in M}$ to a model of T'. By Beth's definability theorem for each $\phi(\bar{x}, \bar{y})$ there is a formula $d\phi(\bar{y})$ with parameters from M that defines P_ϕ.

If f is an automorphism of M fixing E, then f fixes $p_{V,M}$. Thus f fixes $\{\bar{a} \in M : M \models d\phi(\bar{a})\}$. Thus $d\phi$ is equivalent to a formula with parameters from E.

We conclude with an easy observation that is mildly related. The proof is much simpler that the corresponding theorem for differentially closed fields.

Proposition 5.9 If $V \subseteq K^n$ is an irreducible algebraic variety, then V is still irreducible as a σ-closed set.

Proof

Suppose $V = F_1 \cup F_2$ where F_1 and F_2 are σ-closed proper subsets of V. We can find n and Zariski closed subsets $\widehat{F_1}$ and $\widehat{F_2}$ of $\widehat{V} = V \times \sigma(V) \times \cdots \times \sigma^n(V)$ such that if $\nabla : V \to \widehat{V}$ by $\nabla(x) = (x, \sigma(x), \ldots, \sigma^n(x))$, then $F_i = \nabla^{-1}(\widehat{F_i})$. Since $\widehat{F_1}$ and $\widehat{F_2}$ are proper closed subsets of the irreducible variety $\widehat{V}$ they are of lower dimension, so $\widehat{F_1} \cup \widehat{F_2}$ is also a proper closed subset. Our proposition then follows from the following claim.

claim If $F \subset \widehat{V}$ is Zariski closed, then there is $x \in V$ such that $\nabla(x) \in \widehat{V} \setminus F$.

Let $U = V \times \sigma(V) \times \cdots \times \sigma^{n-1}(V)$ and let

$$V_1 = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in U \times \sigma(U) : y_1 = x_2, y_2 = x_3, \ldots, y_{n-1} = x_n\}.$$

The map $\mu(\bar{x}, \bar{y}) = (\bar{x}, y_n)$ is an isomorphism between V_1 and $\widehat{V}$ so V_1 is irreducible and V_1 projects generically onto U and $\sigma(U)$. Let $F_1 = \mu^{-1}(F)$. By the axioms of ACFA there is $\bar{x} \in U$ such that $(\bar{x}, \sigma(\bar{x})) \in V_1 \setminus F_1$. But then $\bar{x} = (x_1, \sigma(x_1), \ldots, \sigma^{n-1}(x_1))$ and $\nabla(x_1) \in \widehat{V} \setminus F$.