§7 Strongly Stationary Types

We begin this section by developing some facts about stationary types in ACFA. The proofs of 7.1-7.5 use only the Independence Theorem and thus work in any simple theory.

Lemma 7.1 Let $A = \mathrm{acl}_\sigma(A) \subset E = \mathrm{acl}_\sigma(E)$. Suppose $p \in S(E)$ and p does not fork over A then p is stationary if and only if $p|A$ is stationary.

Proof

Clearly if $p|A$ is stationary so is p.

Let $F \supset A$. We may assume $E \downarrow_A F$ (moving F if necessary). Let b_1 and b_2 realize nonforking extensions q_1 and q_2 of $p|A$ to F. Let a realize p.

Apply the Indepencence Theorem to $\mathrm{tp}(a, E/A)$, $\mathrm{tp}(b_i, F/A)$ and $\mathrm{tp}(E, F/A)$. We can find c_i such that c_i realizes a nonforking extension of p to EF and c_i realizes a nonforking extension of q_i to EF. Since p is stationary, $\mathrm{tp}(c_1/EF) = \mathrm{tp}(c_2/EF)$. So $q_1 = q_2$.

Lemma 7.2 Suppose $E = \mathrm{acl}_\sigma(E)$, $\mathrm{tp}(a/E)$ is stationary and $b \in \mathrm{acl}_\sigma(Ea)$. Then $\mathrm{tp}(ab/E)$ is stationary.

Proof

Suppose $F \supset E$, $a \downarrow_E F$ and $\mathrm{tp}(ab/E)$ has at least two nonforking extensions to E. Thus the multiplicity of b over Fa is less that the multiplicity of b over Ea. Choose F such that the multiplicity of b over Fa is minimal.

Let $\psi(a, x, d)$ isolate $\mathrm{tp}(b/Fa)$ where $d \in F$. There is b_1 such that $\mathrm{tp}(ab_1/E) = \mathrm{tp}(ab/E)$ and $\neg\psi(ab_1, , d)$.

Choose F', a', b', b_1', d' such that $\mathrm{tp}(F', a', b', b_1', d'/E) = \mathrm{tp}(F, a, b, b_1, d/E)$ and $F \downarrow_E F_1$.

Apply the Independece Theorem to $\mathrm{tp}(ab, F/E)$, $\mathrm{tp}(a', b', F'/E)$ and $\mathrm{tp}(F, F'/E)$. We can find a nonforking extension q_1 of $\mathrm{tp}(ab/F)$ to $F \cup F'$ such that

$$\psi(x, y, d) \wedge \psi(x, y, d') \in q_1.$$

On the other hand, if we apply the independence theorem to $\mathrm{tp}(ab, F/E)$, $\mathrm{tp}(a', b_1', F'/E)$ and $\mathrm{tp}(F, F'/E)$, we get a nonforking extension q_2 of $\mathrm{tp}(ab/F)$ such that

$$\psi(x, y, d) \wedge \neg\psi(x, y, d') \in q_2.$$

Thus we have lowered the multiplicity, contradicting the assumption of minimality.

We now look at stationary types p with $\mathrm{SU}(p) = 1$. The first observation is that $\mathrm{U}(p) = 1$ and p is a minimal type.

Corollary 7.3 If $E = \mathrm{acl}_\sigma(E)$, $p \in S(E)$ is stationary, $\mathrm{SU}(p) = 1$ and c is a tuple of realizations of p, then $\mathrm{tp}(c/E)$ is stationary.

Proof

Since $\mathrm{SU}(p)=1$ we may assume $c=(\bar{a},\bar{b})$ where $\bar{a}$ is a sequence of independent realizations of p and $\bar{b}\in \mathrm{acl}_\sigma(E\bar{a})$. Clearly $\mathrm{tp}(\bar{a}/E)$ is stationary. Thus $\mathrm{tp}(c/E)$ is stationary by 7.2.

Corollary 7.4 Let $E=\mathrm{acl}_\sigma(E)$ and $p\in S(E)$ be stationary with $\mathrm{SU}(p)=1$. Let M be a $|E|^+$-saturated model, P the set of realizations of p in M, and c a tuple of realizations of p. Then $\mathrm{tp}(c/\mathrm{acl}_\sigma(EP))\vdash \mathrm{tp}(c/M)$.

Proof

By 7.3 $\mathrm{tp}(c/\mathrm{acl}_\sigma(EP))$ is stationary. Thus it is enough to prove that $c\downarrow_{EP}M$. By saturation $\mathrm{tp}(c/M)$ is finitely satisfiable in P. Thus if $\bar{m}\in M$ is algebraically dependent over $\langle EPc\rangle_\sigma$, $\bar{m}$ is algebraically dependent over $\langle EP\rangle_\sigma$ as desired.

Definition Let P be the set of all realizations of $p\in S(E)$. We say that P is *stably embedded* in M if and only if for every definable set $D\subseteq M^n$, there is D_1 definable with parameters from $P\cup E$ such that $D\cap P^n=D_1\cap P$.

Corollary 7.5 If p is a stationary type with $\mathrm{SU}(p)=1$, then p is stably embedded.

Proof

Let $x=(x_1,\ldots,x_n)$. Let $\phi(x,b)$ define D. Let M be a small $|E|^+$-saturated model containing Eb and let P_0 be the realizations of p in M. Let $\Psi=\{\psi(x):\psi(x)\to\phi(x,b)$ and ψ has parameters from $\mathrm{acl}_\sigma(EP_0)\}$. By 7.4 and compactness

$$p(x_1)\wedge\ldots\wedge p(x_n)\wedge\phi(x,b)\vdash\bigvee_{\psi\in\Psi}\psi(x).$$

Thus there is a formula $\theta(v)\in p$ such $\psi_1,\ldots,\psi_m\in\Psi$ that

$$\theta(x_1)\wedge\ldots\wedge\theta(x_n)\wedge(\psi_1(x)\vee\ldots\vee\psi_m(x))$$

and $\phi(x,b)$ agree on P^n. Thus we may assume D is definable over $\mathrm{acl}_\sigma(EP_0)$.

Let $D=D_1,\ldots,D_m$ be the distinct conjugates of D over EP_0. If $D\cap P^n=D_i\cap P^n$ for all i, then $D_0=\bigcap D_i$ is definable over $E\cup P_0$ and $D_0\cap P^n=D\cap P^n$. If not, for each j with $D_j\cap P^n\neq D\cap P^n$ choose $d_j\in(D_j\triangle D)\cap P^n$. It is easy to find D^* definable over $E\cup P_0\cup\{d_2,\ldots,d_m\}$ with $D\cap P^n=D^*\cap P^n$.

We now state an algebraic characterization of stationarity in ACFA.

Proposition 7.6 Let $E=\mathrm{acl}_\sigma(E)$. The following are equivalent.

a) $\mathrm{tp}(a/E)$ is stationary.

b) Suppose $F\supset E$, $F=\mathrm{acl}_\sigma(F)$, $a\downarrow_E F$, $\sigma_1\in Aut(\mathrm{acl}_\sigma(Fa))$, $\sigma_1\supset\sigma|\langle Fa\rangle_\sigma$ and $\sigma_1|\mathrm{acl}_\sigma(Ea)$ is conjugate to $\sigma|\mathrm{acl}_\sigma(Ea)$ under the action of $Gal(\mathrm{acl}_\sigma(Ea)/\langle Ea\rangle_\sigma)$. Then σ and σ_1 are conjugate under the action of $Gal(\mathrm{acl}_\sigma(Fa)/\langle Fa\rangle_\sigma)$.

c) $\mathrm{tp}(\mathrm{acl}_\sigma(Ea)/E)$ is stationary.

d) Suppose $F \supset E$, $F = \mathrm{acl}_\sigma(F)$, $a \underset{E}{\smile\!\!\!\!\!\mid\,} F$, $\sigma_1 \in Aut(\mathrm{acl}_\sigma(Fa))$, $\sigma_1 \supset \sigma | F\mathrm{acl}_\sigma(Ea)$. Then σ and σ_1 are conjugate under the action of $Gal(\mathrm{acl}_\sigma(Fa)/F\mathrm{acl}_\sigma(Ea))$.

Proof

The equivalence of a) and c) follows from 7.2.

The equivalence of a) and b) follows from the following two facts:

i) If $a \underset{E}{\smile\!\!\!\!\!\mid\,} F$, then the quantifier free type of a over F is determined by the quantifier free type of a over E as $\mathrm{tp}^-(a, \sigma(a), \ldots, \sigma^n(a)/F)$ is the heir of $\mathrm{tp}^-(a, \sigma(a), \ldots, \sigma^n(a)/E)$.

ii) $\mathrm{tp}(a/E)$ is determined by the quantifier free type of a over E and the conjugacy class of $\sigma|\mathrm{acl}_\sigma(Ea)$ under $Gal(\mathrm{acl}_\sigma(Ea)/\langle Ea\rangle_\sigma)$.

The equivalence of c) and d) follows from the equivalence of a) and b).

We give a useful strengthening of the notion of stationarity motivated by lemma 5.4, which we recall (giving the proper statement).

Lemma 5.4 Let E be a field and $\widetilde{E}$ be the algebraic closure of E. Suppose there is no finite σ-invariant Galois extension of E. If $\tau \in Aut(\widetilde{E})$ and $\tau \supset \sigma|E$ then τ is conjugate to σ under the action of the Galois group $Gal(\widetilde{E}/E)$.

Definition Suppose $E = \mathrm{acl}_\sigma(E)$. We say that $p \in S(E)$ is *strongly stationary* iff for any a realizing p and any $F \supset E$ with $\mathrm{acl}_\sigma(F) = F$ and $a \underset{E}{\smile\!\!\!\!\!\mid\,} F$, then $F\mathrm{acl}_\sigma(Ea)$ has no proper finite σ-invariant Galois extensions. (Chatzidakis and Hrushovski call this notion *superficially stable.*)

Remark It is an easy exercise in Galois theory to see that if $\sigma(F) = F$, then F has a proper finite σ-invariant Galois extension if and only if F has a proper finite σ-invariant separable extension.

Corollary 7.7 If $E = \mathrm{acl}_\sigma(E)$ and $p \in S(E)$ is strongly stationary, then p is stationary.

Proof By 5.4 and 7.6.

Remark In fact all finite rank stationary types are strongly stationary, but this will come out later.

We next give a useful characterization of strongly stationary types. Let $K \models$ ACFA. For $k \geq 1$, let $M_k = (K, \sigma^k)$.

Lemma 7.8 If $(K, \sigma) \models$ ACFA, then $M_k \models ACFA$.

Proof

It suffices to show that if U and V are irreducible varietes, $V \subset U \times \sigma^k(U)$ and $W \subset V$ is Zariski closd then there is $a \in U$ such that $(a, \sigma^k(a) \in V \setminus U$. Let $V_1 = \{(x, x_1, \ldots, x_k) \in U \times \sigma(U) \times \cdots \times \sigma^k(U) : (x, x_k) \in V\}$ and $W_1 = \{(\bar{x} \in V_1 : (x, x_k) \in W\}$. Then V_1 is irreducible and, as in 5.9 we can find $a \in U$ such that $(a, \sigma(a), \ldots, \sigma^k(a)) \in V_1 \setminus W_1$. Then $(a, \sigma^k(a)) \in V \setminus W$.

If $E = \mathrm{acl}_\sigma(E)$ and $p \in S(E)$, we let $p[k]$ be the reduct of p to M_k.

Propositon 7.9 The following are equivalent.
a) p is strongly stationary.
b) $p[k]$ is stationary or all $k \geq 1$.

Proof
a)$\Rightarrow$ b) Suppose not. Then, by 7.6, there is a a realization of p, and $F = \mathrm{acl}_{\sigma^k}(F)$ such that F and $E_1 = \mathrm{acl}_{\sigma^k}(Ea)$ and F are algebraically disjoint over E and FE_1 has a finite σ^k-invariant Galois extension L.

claim Without loss of generality, we may assume $\mathrm{acl}_\sigma(E_1) \mathop{\smile\!\!\!\!\!\!|}_E \mathrm{acl}_\sigma(F)$.

Let $E_2 = \mathrm{acl}_\sigma(Ea)$ and $F_2 = \mathrm{acl}_\sigma(F)$. Now work in M_k. Choose E_2' such that $\mathrm{tp}(E_2/E_1) = \mathrm{tp}(E_2'/E_1)$ and $E_2' \mathop{\smile\!\!\!\!\!\!|}_{E_1} F$. Let F_2' realize the nonforking extension of $\mathrm{tp}(F_2/F)$ to FE_2'. Then F_2' and E_2' are algebraically disjoint over E.

Let $\tau \in Aut(E_2')$ be the automorphism induced by σ under the isomorphism between E_2 and E_2' over E_1. Similarly let $\rho \in Aut(F_2')$ be the image of σ under the isomorphism between F_2' and F_2. Then $\tau|E_1 = \sigma|E_1$ and $\tau^k = \sigma^k$ on E_2', while $\rho|F_1 = \sigma|F_1$ and $\tau^k = \sigma^k$ on F_2'. Since F_2' and E_2' are algebraically disjoint over E, by 1.4 we can find an automorphism σ_1 of $\mathrm{acl}(E_2'F_2')$ which extends τ and ρ and $\sigma_1^k = \sigma^k$ on $E_2'F_2'$. By 1.2 we can extend this to $(K, \sigma_1) \models$ ACFA. From the point of view of this model $E_2' = \mathrm{acl}_{\sigma_1}(E_1)$ and $F_2' = \mathrm{acl}_{\sigma_1}(F)$. While we haven't specified how σ_1 acts L it is still σ^k-invariant [if L is the splitting field of $f(X)$ over E_1F then it is also the splitting field of $\sigma^k(f(X))$], so L is σ_1^k-invariant. We can move this all back to our original model.

Let $K = \mathrm{acl}_\sigma(E_1)\mathrm{acl}_\sigma(F_2)$ and $L' = KL$. By linear disjointness the minimal polynomial of L over E_1F_2 is still the minimal polynomial of L' over K and L' is σ^k-invariant. Let $L_i' = \sigma^i(L')$ for $i < k$ and let L^* be the composite of $L', L_1', \ldots, L_{k-1}'$. Then L^* is a σ-invariant finite Galois extension of K. This contradicts the fact that p is strongly stationary.

b)$\Rightarrow$ a) Suppose a realises p, $F \supset E$, $\mathrm{acl}_\sigma(F) = F$, $a \mathop{\smile\!\!\!\!\!\!|}_E F$. Let $K = F\mathrm{acl}_\sigma(Ea)$ and suppose L/K is a finite σ-invariant Galois extension. If $\tau \in Gal(L/K)$, then $\sigma^{-1}\tau\sigma \in Gal(L/K)$. So σ acts on $Gal(L/K)$ be conjugation. Since $Gal(L/K)$ is finite, there is a k such that the action of σ^k on $Gal(L/K)$ is trivial, ie. $\sigma^k|L$ commutes with every element of $Gal(L/K)$. Let $\rho \in Aut(\widetilde{K})$ such that $\rho|K = \sigma^k$, $\rho|L \neq L$ and $\tau(L) = L$. It follows from 5.3 that there are distinct nonforking extensions of $p[k]$ to E.

Corollary 7.10 Let $A = \mathrm{acl}_\sigma(A) \subset E = \mathrm{acl}_\sigma(E)$.

a) Suppose $p \in S(E)$ and p does not fork over A then p is strongly stationary if and only if $p|A$ is strongly stationary.

b) If $p \in S(E)$ is strongly stationary and $\bar{c}$ is a tuple of realizations of p, then $\mathrm{tp}(\bar{c}/E)$ is strongly stationary.

Proof Clear from 7.9, 7.1 and 7.2.

We now start moving towards the Dichotomy theorem. We first show that strongly stationary types are orthogonal to the fixed field and then that they are locally modular.

Theorem 7.11 Let $E = \mathrm{acl}_\sigma(E)$. If $p \in S(E)$ is strongly stationary, then $p \perp \sigma(x) = x$.

Proof

Suppose $p \not\perp \sigma(x) = x$. Then we can find $F \supseteq E$ with $\mathrm{acl}_\sigma(F) = F$, a realizing p such that $a \mathop{\smile\!\!\!\!\!\mid}_E F$, and $d \in \mathrm{Fix}(\sigma)$ such that $a \not\mathop{\smile\!\!\!\!\!\mid}_F d$.

claim There is $b \in \mathrm{Fix}(\sigma) \cap \langle Fa\rangle_\sigma \setminus F$.

As $\mathrm{SU}(\sigma(x) = x) = 1$, $d \in \mathrm{acl}_\sigma(F, a)$. So d is algebraic over $\langle Fa\rangle_\sigma$. Let

$$f(X) = X^n + b_{n-1}X^{n-1} + \ldots + b_0$$

be the minimal polynomial of d over $\langle Fa\rangle_\sigma$. Since $\sigma(d) = d$, $d^n + \sum_{i=0}^{n-1} \sigma(b_i)d^i = 0$. But then we must have $\sigma(b_i) = b_i$ for $i = 1, ...n$. Since $d \notin F$, some $b_i \notin F$.

Let $c \in \mathrm{Fix}(\sigma)$ with $c \mathop{\smile\!\!\!\!\!\mid}_F a$. Let $F' = \mathrm{acl}_\sigma(Fc)$. By transitivity, $a \mathop{\smile\!\!\!\!\!\mid}_E F'$. Let $K = F'\mathrm{acl}_\sigma(Ea)$. Clearly $K(\sqrt{b+c})$ is a σ-invariant Galois extension of K (use $\sqrt[3]{b+c}$ in characteristic 2). The claim below says that it is a proper extension and this contradicts strong stationarity.

Fact Suppose E is algebraically closed, F and K are algebracically disjoint over E, $x \in F \setminus E$ and $y \in K \setminus E$, then $\sqrt{x+y} \notin FK$.

Without loss of generality $F = \mathrm{acl}(E(x))$ and $K = \mathrm{acl}(E(y))$. There is a rational function $f(\bar{X}, Y) \in F(\bar{X}, Y)$ and $\bar{\alpha} \in K$ such that $f(\bar{\alpha}, y)^2 = x + y$. Suppose $\psi(\bar{u}, v)$ is an $\mathcal{L}^-$-formula with parameters in E such that $\psi(\bar{u}, y)$ isolates $\bar{\alpha}$ over $E(y)$. Let $\phi(v)$ be the formula

$$\exists \bar{u}\ \psi(\bar{u}, v) \wedge f(\bar{u}, v)^2 = x + v.$$

Since $\phi(\bar{y})$ holds and y is transcendetal over F, ϕ holds for almost all elements of F. Let L be a finite extension of $E(x)$ such that $f \in L(\bar{X}, Y)$. Then for all but finitely many $a \in E$, there is $\bar{\alpha} \in E$ such that $f(\bar{\alpha}, a)^2 = x + a$. Thus $x + a$ has a square root in L for almost all $a \in E$. But the $x + a$ are prime in $E[X]$ and only finitely many primes of $E[X]$ ramify in L, a contradiction.

We next review some facts about local modularity. Though the following definition and theorem are well known for stable theories, they work equally well in arbitrary theories if p is stably embedded. We refer to Pillay's *Geometric Stability Theory.*

Definition (T stable) Let $p \in S(A)$ be stationary with $\mathrm{U}(p) = 1$. We say p is *locally modular* if there is $B \supseteq A$ and $p_1 \in S(B)$ a nonforking extension of p such that if a and b realize p_1 and C is a finite set of realziations of p_1 and $b \in \mathrm{acl}(BCa)$, then there is $c \in \mathrm{acl}_\sigma(BC)$ such that $b \in \mathrm{acl}(Bca)$.

Theorem 7.12 (T stable) 1) Let $p \in S(A)$ be stationary with $\mathrm{U}(p) = 1$. Then p is locally modular iff whenever a_1, a_2 realize p, $B \supseteq A$ is algebraically closed in M^{eq}, $\mathrm{U}(a_1, a_2/) = 1$, and c is the canonical base for $\mathrm{tp}(a_1, a_2/B)$, then $SU(c/A) = 1$.

2) (Buechler) If $p \in S(A)$ is stationary with $\mathrm{U}(p) = 1$ and p nonlocally modular. Then for every $k \geq 2$, there are a_1, a_2 and an algebraically closed $B \supseteq A$, such that the canonical base for $\mathrm{tp}(a_1, a_2/B)$ has rank k over B.

Warning: This theorem is true in all characteristics but I will use characteristic zero at some point.

Theorem 7.13 Let $E = \mathrm{acl}_\sigma(E)$. Let $p \in S(E)$ be strongly stationary with $\mathrm{SU}(p) = 1$. Then p is locally modular.

Proof

If $p = \mathrm{tp}(\alpha/E)$, then by replacing p by $\mathrm{tp}(\alpha, \ldots, \sigma^k(\alpha))$ for some k we may assume that

$$\mathrm{acl}_\sigma(E\bar{\beta}) = \mathrm{acl}(E\bar{\beta}) = \mathrm{acl}(E\sigma(\bar{\beta})) \tag{$*$}$$

for any tuple of realizations $\bar{\beta}$ of p. (Note: Since strong stationarity of $\mathrm{tp}(a/E)$ is a property of $\mathrm{tp}(acl(Ea)/E)$ this is no problem.)

Suppose x, y are realizations of p, independent over E, $a_1 = (x, y)$ $E \subseteq F = \mathrm{acl}_\sigma(F)$, $q = \mathrm{tp}(a_1/F)$, $\mathrm{SU}(q) = 1$ and the canonical base γ for q has rank 3 over A. To simplify things we will assume that there is a tuple of realizations d_1 of p such that $\gamma \in \mathrm{dcl}(Ed_1)$ and $d_1 \in \mathrm{acl}_\sigma(\gamma)$. (We may have to expand E to insure this.)

Let a_2 realize q with $a_1 \mathop{\smile\!\!\!\!\!\!|}_{Ed_1} a_2$.

claim 1 $a_1 \mathop{\smile\!\!\!\!\!\!|}_E a_2$.

It suffices to show $\mathrm{SU}(a_2/Ea_1) = 2$. Suppose not. Since $\mathrm{SU}(a_2/Ea_1d_1) = 1$, we must have $\mathrm{SU}(a_2/Ea_1)$. But then q does not fork over Ea_1 and $d_1 \in \mathrm{acl}_\sigma(Ea_1)$ and $\mathrm{SU}(d_1/E) \leq 2$ a contradiction.

claim 2 $\mathrm{SU}(d_1/Ea_1a_2) = 1$

Since a_1 and a_2 are realizations of q independent over d_1,

$$\mathrm{SU}(a_1a_2d_1/E) = \mathrm{SU}(a_1, a_2/Ed_1) + \mathrm{SU}(d_1/E) = 2 + 3 = 5.$$

So

$$5 = \mathrm{SU}(a_1a_2d_1/E) = \mathrm{SU}(d_1/Ea_1a_2) + \mathrm{SU}(a_1a_2/E) = \mathrm{SU}(d_1/Ea_1a_2) + 4.$$

Choose d_2 realizing $\mathrm{tp}(d_1/\mathrm{acl}_\sigma(Ea_1a_2))$ with $d_1 \mathop{\smile\!\!\!\!\!\!|}_{Ea_1a_2} d_2$. By claim 2, d_1 and d_2 are not interalgebraic over E, thus they are canonical bases for different rank one types containing a_i. Thus $a_1, a_2 \in \mathrm{acl}(Ed_1d_2)$.

claim 3 $d_1 \mathop{\smile\!\!\!\!\!\!\!\;\,\,|}_{E} d_2$

$$\begin{aligned} 6 &= \mathrm{SU}(d_2/d_1a_1a_2E) + \mathrm{SU}(d_1a_1a_2/E) \\ &= \mathrm{SU}(d_1d_2a_1a_2/E) \\ &= \mathrm{SU}(d_1d_2/E) \\ &= \mathrm{SU}(d_2/Ed_1) + \mathrm{SU}(d_1/E) \\ &\leq \mathrm{SU}(d_2/E) + \mathrm{SU}(d_1/E) = 3+3 \end{aligned}$$

Since $\mathrm{SU}(d_1/E) = 3$, $\mathrm{SU}(d_2/Ed_1) = 3$.

claim 4 $\mathrm{tp}(a_1/Ed_1d_2) = \mathrm{tp}(a_2/Ed_1d_2)$

By claim 1, $\mathrm{tp}(a_2a_1/d_1E) = \mathrm{tp}(a_1a_2/d_1E)$ and then, by choice of d_2, $\mathrm{tp}(a_2a_1/d_2E) = \mathrm{tp}(a_1a_2/d_2E)$. Since d_1 and d_2 are realizations of the same strong type over Ea_1a_2, $\mathrm{tp}(a_2a_1d_1d_2/E) = \mathrm{tp}(a_1a_2d_1d_2/E)$. In particular $\mathrm{tp}(a_1/Ed_1d_2) = \mathrm{tp}(a_2/Ed_1d_2)$.

Let $E_i = \mathrm{acl}_\sigma(Ed_i)$. (Recall that by $(*)$ $E_i = \mathrm{acl}(Ed_i)$). Let $K = E_1E_2$. By 7.10 $\mathrm{tp}(d_i/E)$ is strongly stationary, thus K has no proper σ-invariant finite Galois extensions.

Let $c_1, \ldots, c_n$ be conjugates of a_1 over K which are (field theoretically) interalgebraic with a_1 over E. Let e be the canonical parameter for $\{c_1, \ldots, c_n\}$.

claim 5 $e \in K(\sigma(e))$

Note that $\mathrm{acl}(Ec_i) = \mathrm{acl}(Ee) = \mathrm{acl}(Ea_1)$ for any $i \leq n$. Thus

$$\mathrm{acl}_\sigma(Ee) = \mathrm{acl}(Ee) = \mathrm{acl}(E\sigma(e))$$

by $(*)$. In particular e and $\sigma(e)$ are interalgebraic over E.

Suppose $e \notin K(\sigma(e))$. Choose $e' \neq e$ such that $\mathrm{tp}^-(e/K\sigma(e)) = \mathrm{tp}^-(e'/K\sigma(e))$ (*Here I am using characteristic 0*). Then e' and $\sigma(e)$ are interalgebraic over E, so e' and e are interalgebraic over E. But e' is the canonical parameter for $\{c'_1, \ldots, c'_n\}$ and then

$$\mathrm{acl}(Ec'_i) = \mathrm{acl}(Ee') = \mathrm{acl}(Ee) = \mathrm{acl}(Ec_j)$$

for all $i, j \leq n$. By the definintion of e each $c'_i \in e$. Thus $e = e'$, a contradiction.

Thus $K(e)$ is a σ-invariant finite extension of K. Since p (and hence $\mathrm{tp}(d_1/E)$) is strongly stationary, $e \in K$ (*Here I am perhaps using characteristic 0 to get $K(e)/K$ separable.*). Since a_1 and a_2 realize the same type over K, $a_2 \in e$ and a_1 and a_2 are interalgebraic over E, but this contradicts claim 1.

Remark The usual arguments suffice to show that the fixed field in not 1-based and no nonalgebriac type in the fixied field is locally modular. We can develop enough geometric stability theory in simple theories to insure that any SU-rank 1 type nonorthogonal to a locally modular type is itself locally modular. This gives a different proof of 7.11.