§8 Degree one types

Let $E = \mathrm{acl}_\sigma(E)$. In this section we will be look at $\mathrm{tp}(a/E)$ where $\deg_\sigma(a/E) = 1$. Note that in this case $\mathrm{acl}(Ea) = \mathrm{acl}_\sigma(Ea)$. We let $\mathrm{mult}(b/F)$ denote the field theoretic multiplicity of b over F.

Warning: From this point on we will assume that we are in characteristic zero.

Definition Let $E = \mathrm{acl}_\sigma(E)$. Let $p = \mathrm{tp}(a/E)$. We say p is *bounded* iff $\mathrm{acl}(Ea) = \mathrm{acl}_\sigma(Ea)$ and there is a number k such that $\mathrm{mult}(\sigma^i(a)/E(a)) \leq k$ for all $i \in \mathbf{Z}$.

We will prove that bounded degree one types are nonorthogonal to the fixed field.

Lemma 8.1 Suppose $\mathrm{tp}(a/E)$ is bounded and $\mathrm{acl}(Eb) = \mathrm{acl}(Ea)$.

a) $\mathrm{tp}(b/E)$ is bounded.

b) If $\sigma(b) \in E(b)$, then $E(b) = E(\sigma(b))$.

Proof

a) Clearly $\mathrm{acl}_\sigma(Eb) = \mathrm{acl}(Eb)$. Let $l = \mathrm{mult}(b/Ea)$ and $m = \mathrm{mult}(a/Eb)$. Then

$$\mathrm{mult}(\sigma^i(b)/Eb) \leq \mathrm{mult}(\sigma^i(b)/E\sigma^i(a)) \cdot \mathrm{mult}(\sigma^i(a)/Ea) \cdot \mathrm{mult}(a/Eb) = lkm,$$

since $\mathrm{mult}(\sigma^i(b)/E\sigma^i(a)) = \mathrm{mult}(b/Ea) = l$. Thus $\mathrm{tp}(b/E)$ is bounded.

b) Suppose $l = \mathrm{mult}(b/E\sigma(b))$. Then $\mathrm{mult}(\sigma^{-n}(b)/E\sigma^{1-n}(b)) = l$, for all n. We claim that $\mathrm{mult}(\sigma^{-n}(b)/Eb) = l^n$ for all $n > 0$.

Since $\sigma(b) \in E(b)$, $E(\sigma^n(b)) \supset E(\sigma^{1-n}(b)) \supset E(b)$. Thus

$$\begin{aligned}\mathrm{mult}(\sigma^{-n}(b)/Eb) &= \mathrm{mult}(\sigma^{-n}(b), \sigma^{1-n}(b)/Eb) \\ &= \mathrm{mult}(\sigma^{-n}(b)/E\sigma^{1-n}(b)) \cdot \mathrm{mult}(\sigma^{1-n}(b)/Eb) = l(l^{n-1})\end{aligned}$$

by induction.

Lemma 8.2 Suppose $\mathrm{tp}(a/E)$ is bounded. There is $b \in \mathrm{acl}(Ea)$ such that $E(b) = E(\sigma(b))$ and $\mathrm{acl}(Eb) = \mathrm{acl}_\sigma(E(a))$.

Proof

Replacing a by $(a, \sigma(a), \ldots, \sigma^m(a))$ for sufficiently large m, we may assume

$$\mathrm{mult}(a/E\sigma(a)) = \mathrm{mult}(a/E\sigma(a), \sigma^2(a) \ldots). \qquad (*)$$

By the 8.1 a), the new type is still bounded.

claim $\mathrm{mult}(\sigma^n(a)/Ea\sigma(a)) = \mathrm{mult}(\sigma^n(a)/E\sigma(a))$ for any $n > 0$

$$\mathrm{mult}(a, \sigma^n(a)/E\sigma(a)) = \mathrm{mult}(a/E\sigma(a)\sigma^n(a)) \cdot \mathrm{mult}(\sigma^n(a)/E\sigma(a))$$

and

$$\mathrm{mult}(a, \sigma^n(a)/E\sigma(a)) = \mathrm{mult}(\sigma^n(a)/Ea\sigma(a)) \cdot \mathrm{mult}(a/E\sigma(a)).$$

By $(*)$, $\mathrm{mult}(a/E\sigma(a)) = \mathrm{mult}(a/E\sigma(a)\sigma^n(a))$, the claim follows.

In particular $\mathrm{tp}^-(\sigma^n(a)/Ea\sigma(a))$ and $\mathrm{tp}^-(\sigma^n(a)/E\sigma(a))$ have the same realizations.

claim There is $k < 0$ such that $\mathrm{tp}^-(a/E\sigma^m(a))$ and $\mathrm{tp}^-(a/E\sigma^k(a))$ have the same realizations for all $m < k$.

By the previous claim (after applying σ^{-n})

$$\mathrm{mult}(a/E\sigma^{-n}(a)) \geq \mathrm{mult}(a/E\sigma^{-n}(a), \sigma^{1-n}(a)) = \mathrm{mult}(a/E\sigma^{1-n}(a)).$$

Moreover the set of realizations of $\mathrm{tp}^-(a/E\sigma^{-n}(a))$ contains the realizations of $\mathrm{tp}^-(a/E\sigma^{1-n}(a))$. By boundedness, we can choose $k < 0$ such that $\mathrm{mult}(a/E\sigma^k(a))$ is maximal. Thus $\mathrm{tp}^-(a/E\sigma^m(a))$ and $\mathrm{tp}^-(a/E\sigma^k(a))$ have the same realizations for $m < k$.

Let k be as in the second claim. Let $\{a_1, \ldots, a_m\}$ be the realizations of $\mathrm{tp}(a/E(\sigma^k(a))$ and let e be the canonical parameter for $\{a_1, \ldots, a_m\}$. Clearly $\mathrm{acl}(Ee) = \mathrm{acl}(Ea)$ (recall $\mathrm{acl}(Ea) = \mathrm{acl}(E\sigma^l(a)) = \mathrm{acl}_\sigma(Ea)$ for any l). Clearly $e \in E(\sigma^l(a))$ for $l < m$ as it is definable over $E(\sigma^l(a))$. Let $F = \bigcap\{E(\sigma^l(a)) : l \leq m\}$. Clearly $\sigma(F) \subseteq F$. Let $F = E(b)$ for some finite tuple b. Then $\sigma(b) \in E(b)$. Clearly

$$\mathrm{acl}(Ea) \supseteq \mathrm{acl}(F) \supseteq \mathrm{acl}(Ee) = \mathrm{acl}(Ea).$$

Thus by 8.1, $E(b) = E(\sigma(b))$.

Before proving the main result of this section we review a bit of the theory of divisors on curves. Let C be curve. A *divisor* on C is a formal sum $\displaystyle\sum_{i=1}^{n} m_i p_i$ where $m_i \in \mathbf{Z}$ and $p_i \in C$. If f is a rational function on C, then (f) is the divisior $\sum_{i=1}^n k_i p_i - \sum_{i=1}^n l_i q_i$, where $\{p_1, \ldots, p_n\}$ are the zeros of f, p_i is a zero of multiplicity k_i, $\{q_1, \ldots, q_m\}$ are the poles of f and l_i is the multiplicity of the pole q_i.

If $D_0 = \sum m_i p_i$ and $D_1 = \sum n_i p_i$, we say $D_0 \geq D_1$ if $m_i \geq n_i$ for all i. Let $L(D)$ be the set of all rational function on C such that $(f) \geq -D$. If $f_0, \ldots, f_n$ is a basis for $L(D)$ where $n > 0$ we define a rational map $\phi : C \to \mathbf{P}^n$ by $\phi(p) = (f_0(p) : \ldots : f_n(p))$. At first this only makes sense if each f_i is defined at p and some $f_i(p) \neq 0$. Choose z a local coordinate at p. Let m_i be such that $f_i = z^{m_i} g_i$ where $g_i(p) \neq 0$. Let $m = \min m_i$. Let $h_i = z^{m_i - m} g_i$. For x near p,

$$(f_0(x) : \ldots : f_n(x)) = h_0(x) : \ldots : h_n(x)).$$

All of the $h_i(p)$ are defined and not all are zero. Thus we can extend ϕ to p by $\phi(p) = (h_0(p) : \ldots : h_n(p))$. So ϕ extends to a morphism from C into $\P^n$.

We say the divisor is *very ample* if ϕ is one to one and the image is a smooth curve. The following lemma is a consequence of the Riemann-Roch Theorem.

Lemma 8.3 If $D > 0$, then $L(D)$ is finite dimensional and there is a number m such that mD is very ample.

We are now ready to prove the main theorem of this section.

Theorem 8.4 Suppose $E \models$ ACFA, $\deg_\sigma(a/E) = 1$ and $\mathrm{tp}(a/E)$ is bounded, then there is c in the fixed field such that $\mathrm{acl}(Ec) = \mathrm{acl}(Ea)$.

Proof

Let b be as in lemma 8.2. Since $\deg_\sigma(a/E) = 1$, $E(b)$ is the function field of a smooth projective curve X. We think of b as a generic point of this curve. Then $\sigma(X)$ is a smooth projective curve and $\sigma(b)$ is a generic point of $\sigma(X)$. Since $E(b) = E(\sigma(b))$, the curves X and $\sigma(X)$ are birationally equivalent. Any birationally map between smooth projective curves extends to an isomorphism. Thus there is an isomorphism $g : X \to \sigma(X)$ with $g(b) = \sigma(b)$.

Since $E \models$ ACFA, there is $\alpha \in X(E)$ with $g(\alpha) = \sigma(\alpha)$. Consider the divisor $D = \alpha$. By 8.3 there is an m such that $m\alpha$ is very ample. Let $f_0, \ldots, f_n$ be a basis for $L(mD)$. Let $\phi(x) = (f_0(x) : \ldots : f_n(x))$ be the embedding in to $\mathbf{P}^n$. The functions $\sigma(f_0), \ldots, \sigma(f_n)$ give the embedding $\sigma(\phi) : \sigma(X) \to \mathbf{P}^n$. Thus $\sigma(f_0) \circ g, \ldots, \sigma(f_n) \circ g$ determine a second isomorphic embedding of X into $\mathbf{P}^n$. Clearly $\sigma(f_i) \in L(m\sigma(\alpha)) = L(mg(\alpha))$ and $\sigma(f_0), \ldots, \sigma(f_n)$ is a basis for $L(m\sigma(\alpha))$. Thus $\sigma(f_0) \circ g, \ldots, \sigma(f_n) \circ g$ is also a basis for $L(mD)$. Thus there is $A \in PGL_n(E)$ such that $A\phi = \sigma(\phi) \circ g$.

Let $V = \{(B, C) \in PGL_n(E) \times PGL_n(E) : B = CA\}$ Since $E \models$ ACFA, there is $B \in PGL_n(E)$ such that $(B, \sigma(B)) \in V$. Thus $B = \sigma(B)A$.

Let $\psi = B\phi : X \to \mathbf{P}^n$. Note that

$$\psi = \sigma(B)A\phi = \sigma(B)\sigma(\phi) \circ g = \sigma(B\phi) \circ g = \sigma(\psi) \circ g.$$

Let $c = \psi(b)$. Then $\mathrm{acl}(Eb) = \mathrm{acl}(Ec)$ and

$$\sigma(c) = \sigma(\psi(b)) = \sigma(\psi)(\sigma(b)) = \sigma(\psi) \circ g(b) = \psi(b) = c.$$

Corollary 8.5 Suppose $E = \mathrm{acl}_\sigma(E)$, $p \in S(E)$ and $\mathrm{SU}(p) = 1$. Then $p \not\perp \sigma(x) = x$ if and only if $\deg_\sigma(p) = 1$ and p is bounded.

Proof

Suppose $F \supset E$, $F = \mathrm{acl}_\sigma(F)$ and p_1 is the nonforking extension of p to F. It is easy to see that $\deg_\sigma(p) = 1$ if and only if $\deg_\sigma(p_1) = 1$ and if $\deg_\sigma(p) = 1$, then p is bounded if and only if p_1 is bounded.

($\Rightarrow$) If $p \not\perp \sigma(x) = x$, we can find F, a, and b such that $F = \mathrm{acl}_\sigma(F) \supseteq E$, a realizes p, $a \underset{E}{\perp\!\!\!\perp} F$, $\sigma(b) = b$ and $a \underset{F}{\not\perp\!\!\!\perp} b$. Then $a \in \mathrm{acl}_\sigma(Fb)$, so $\deg_\sigma(a/F) = 1$. Moreover since $\mathrm{tp}(b/F)$ is bounded and $\mathrm{acl}(Fa) = \mathrm{acl}_\sigma(Fa) = \mathrm{acl}_\sigma(Fb) = \mathrm{acl}(Fb)$, by 8.1a) $\mathrm{tp}(a/F)$ is bounded. By the above remark $\mathrm{tp}(a/E)$ is bounded.

($\Leftarrow$) We may assume E is a model and appeal to 8.4.

Suppose F is an algebraically closed field and L/F is a finitely generated extension of transcendence degree one over F. We call L a *function field* over F. There is a unique smooth projective curve C defined over F such that L is the function field of C. For $p \in C$, let O_p be the ring of functions in L which are defined at p. Then O_p is a valuation ring for L. If z is a local coordinate at p, the valuation is given by $v(f) = n$, where $f = z^n g$ where g is a unit in O_p. There is a one to one correspondence between nontrivial valuation subrings of L containing F and points on C. (We will only consider valuations which extend F.) If E is a subfield of F we say that a valuation on L is defined over F if the corresponding point on C is an E-rational point.

Suppose L_0 and L_1 are function fields over F and $L_0 \subset L_1$. Let $n = \deg(L_1/L_0)$ and suppose v is a valuation on L_0. By the place extenstion theorem, v extends to a valuation on L_1. There are at most n extensions of v to L_1. We say that *v ramifies in L_1* if there are fewer than n extensions of v to L_1. If u is an extentsion of v to L_1 we also say *u ramifies over L_0*. If L is an infinite algebraic extension of L_0 and v is a valutaion on L_0, we say v ramifies in L if it ramifies in some $L_0 \subset L_1 \subset L$ where L_1/L_0 is finite.

Ramification can be given a geometric meaning. Let C_0 and C_1 be the smooth projective curves corresponding to L_0 and L_1. The inclusion of L_0 into L_1 gives rise to a surjective projection $\pi : C_1 \to C_0$. If $x \in C_0$ is a generic point, then $|\pi^{-1}(x)| = n$. We say π ramifies above $y \in C_0$ if $|\pi^{-1}(y)| < n$.

We give some basic lemmas.

Lemma 8.6 Let F, L_i and C_i be as above.

a) The set of valuations on L_0 which ramify in L_1 is finite.

b) If E is an algebraically closed subfield of F and C_0 and C_1 are defined over E, then all of the points where π ramifies are E-rational.

c) Let E and F be algebraically closed with $E \subset F$. Suppose C is a smooth projective curve defined over E, $E(C)$ is the function field of C and F is algebraically disjoint from $E(C)$ over E. Then any valuation on $FE(C)$ which ramifies in $F\mathrm{acl}(E(C))$ is defined over E.

Proof

a) and b) are immediate. For c), let L be a finite extension of $FE(C)$ where v ramifies. There is L_1 a finite extension of $E(C)$ such that $FL_1 = L$. Let C_1 be the curve defined over E with function field L_1 and let $\pi : C_1 \to C$ be the natural morphism. By c) all of the ramification points of π are E-rational.

Lemma 8.7 Suppose L_0 and L_1 are finite extensions of F, u is a valuation on F, v is an extension of u to L_0 and w is and extension of v to L_0L_1. If v does not ramify over F, then w does not ramify over L_1.

Proof

Let $n = \deg(L_0/F)$ and $m = \deg(L_0L_1/L_1) \leq n$. Let $\sigma_1, \ldots, \sigma_m$ be the automorphisms of L_0L_1/L_1. These restict to distinct automorphsims of L_0/F. Since u does not ramify over F. The valuations $\sigma_i(u)|L_0$ are distinct. Hence the valuations $\sigma_1(u), \ldots, \sigma_m(u)$

are distinct valuations extending $u|L_1$. So u does not ramify over L_1.

The following results is harder and we omit the proof (see Chatzidakis-Hrushovski).

Lemma 8.8 Let $E \subset F$, both algebraically closed. Let C be a smooth projective curve defined over E. Let F be algebraically disjoint from $E(C)$ over F. Suppose L is a finite Galois extension of $F(C)$ not contained in $F\mathrm{acl}(E(C))$. Then there is a valuation on $F(C)$ which ramifies in L and is not defined over E.

The basic (non-trivial) idea is that if $\pi : C_1 \to C_0$ and all of the ramification points of C_1 are E-rational, then C_1 is isomorphic to a curve defined over E.

We are now ready to finish the dicothemy theorem for degree one types.

Theorem 8.9 Suppose $E = \mathrm{acl}_\sigma(E)$, $\deg_\sigma(a/E) = 1$ and $\mathrm{tp}(a/E)$ is not bounded. Then $\mathrm{tp}(a/E)$ is strongly stationary.

Proof

Suppose not. Then there is $F = \mathrm{acl}_\sigma(F)$ with $E \subset F$, $a \mathop{\smile\!\!\!\!\!\!|}_E F$ and L a finite σ-invariant Galois extension of $K = F\mathrm{acl}(Ea)$ (recall that since $\deg_\sigma(a/E) = 1$, $\mathrm{acl}(Ea) = \mathrm{acl}_\sigma(Ea)$). There is L_1 a finite Galois extension of $F(a)$ such that $KL_1 = L$.

Let S be the set of valuations u on L_1 which ramify over $F(a)$ and the restriction of u to $F(a)$ is not defined over E. By 8.6 a) and 8.8, S is finite and non-empty. Since $\mathrm{tp}(a/E)$ is not bounded we can choose $k \in \mathbf{Z}$ such that $\mathrm{mult}(\sigma^k(a)/Fa) > |S| + \deg(L_1/F(a))$.

Choose u in S. Let v be the restriction of u to $F(a)$.

claim 1 u does not ramify in L_2.

Since v is not defined over E, by 8.6 c) it does not ramify in K. Thus by 8.8 u does not ramify in L. Since L is σ-invariant, $L \supset \sigma^k(L_1)$. Thus u does not ramify in L_2.

claim 2 v does not ramify in $F(a, \sigma^k(a))$.

By 8.6 c), every valuation on $F(a)$ which ramifies in $F(a, \sigma^k(a))$ is defined over E.

Let $w_1, \ldots, w_n$ be distinct valuations on L_2 extending u. By claim 1, $n \geq \deg(L_2/L_1)$. Our choice of k, insures that $n > |S|$. Let z_i be the restriction of w_i to $F(\sigma^k(a))$.

claim 3 No z_i is defined over E and no z_i ramifies in $F(a, \sigma^k(a))$.

Look at this geometricly. Suppose C_0, C_1, C_2 are the smooth projective curves corresponding to $F(a, \sigma^k(a))$, $F(a)$ and $F(\sigma^k(a))$, $\pi_i : C_0 \to C_i$ is the corresponding morphism. Since $a \mathop{\smile\!\!\!\!\!\!|}_E F$ these varieties and morphisms are defined over E. Let $x \in C_1$ and $y \in C_2$ be the points corresponding to the valuations v and z_i. Then $y \in \pi_2(\pi_1^{-1}(x))$ and $x \in \pi_1(\pi_2^{-1}(y))$. So x and y are interalgebraic over E. As v is not defined over E, neither is z_i.

As in claim 2, z_i does not ramify in $F(a, \sigma^k(a))$.

claim 4 Each z_i ramifies in L_2.

Let t_i be the restriction of w_i to $F(a, \sigma^k(a))$. Suppose z_i does not ramify in L_2. By claim 3, t_i does not ramify in L_2. But then, by claim 2, w_i does not ramify over $F(a)$. But w_i extends u so this is a contradition.

claim 5 Each z_i ramifies in $\sigma^k(L_1)$.

Since L is σ-invariant, $L = K\sigma^k(L_1)$. By 8.6 c), z_i does not ramify in K. Since z_i ramifies in L, it must ramify in $\sigma^k(L_i)$ by 8.7.

Since all of the w_i restrict to u on L_1, the restrictions to $\sigma^k(L_i)$ are distinct. Since the z_i are not definable over E, each restriction is in $\sigma^k(S)$. But then $|\sigma^k(S)| \geq n > |S|$, a contradiction.

To summarize let us state the dichotemy theorem for degree 1 types.

Corollary 8.10 If $E = \mathrm{acl}_\sigma(E)$ and $\deg_\sigma(a/E) = 1$, then either

i) $\mathrm{tp}(a/E)$ is bounded and non-orthogonal to the fixed field, or

ii) $\mathrm{tp}(a/E)$ is unbounded, strongly stationary and locally modular.