Logarithmic-Exponential Series*

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Goal: To build natural nonstandard models of the theory of $\mathbf{R}_{\exp} = (\mathbf{R}, +, \cdot, <, \exp)$ which reflect asymptotic properties of definable functions.

Definition: An *algebraic function* is a partial function $a_0, \ldots, a_{n-1} \mapsto$ the least solution to $X^n + \sum a_i X^i = 0$. (ie. Skolem functions for real closed fields)

We say $f : (r, +\infty)$ is a *Logarithmic-Exponential function* (LE) if it is a composition of $\exp, \log$ and algebraic functions.

LE-functions are good benchmarks for asymptotic growth. For examply if $p(X, Y) \in \mathbf{R}[X, Y]$ and $y' = p(x, y)$ then y is asymptotic to an LE-function.

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Let $i(x)$ be the compositional inverse to $x \mapsto x \log x$. Then

$$i(x) \sim \frac{x}{\log x}.$$

Question: (Hardy) Is $e^{i(x)}$ asymptotic to an LE-function?

Theorem: a) (Shackell '93) $e^{e^{i(x)}}$ is not asymptotic to an LE-function.

b) (DMM) $e^{i(x)}$ is not asymptotic to an LE-function.

Suppose $\mathcal{M}$ is a nonstandard model of $Th(\mathbf{R}_{\exp})$. If $e^{i(x)} \sim f(x)$, where f is an LE-function, then for any infinite $a \in \mathcal{M}$,

$$e^{i(a)}/f(a) = 1 + \epsilon$$

where ϵ is infinitesimal.

How do we build nonstandard models?

Let $\mathbf{R}_{\mathrm{e}} = (\mathbf{R}, +, \cdot, <, e^x|[0,1])$.

Theorem (Ressayre/DMM) $(M, +, \cdot, <, E)$ is a model of $Th(\mathbf{R}_{\exp})$ if and only if:

E1) $(M, +, \cdot, <, E|[0,1]) \models Th(\mathbf{R}_{\mathrm{e}})$

E2) $E(x+y) = E(x)E(y)$

E3) $x < y \to E(x) < E(y)$

E4)$_n$ $x > n^2 \to E(x) > x^n$

E5) $x > 0 \to \exists y\ E(y) = x$

Suprisingly it is easy to build nonstandard models of $Th(\mathbf{R}_{\mathrm{e}})$.

Let Γ be a divisible ordered abelian group. We consider formal sums

$$f = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma,$$

$a_\gamma \in \mathbf{R}$ for all $\gamma \in \Gamma$. The *support* of f is

$$\mathrm{supp}\ f = \{\gamma : a_\gamma \neq 0\}.$$

Let

$$\mathbf{R}((t^\Gamma)) = \{f : \mathrm{supp}\ f \text{ is well ordered}\}.$$

The operations

$$\sum a_\gamma t^\gamma + \sum b_\gamma t^\gamma = \sum (a_\gamma + b_\gamma) t^\gamma$$

$$\Big(\sum a_\gamma t^\gamma\Big)\Big(\sum b_\gamma t^\gamma\Big) = \sum_{\gamma\in\Gamma}\Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\Big) t^\gamma$$

make $\mathbf{R}((t^\Gamma))$ a field.

There is a natural valuation

$$v\Big(\sum a_\gamma t^\gamma\Big) = \text{least } \gamma \text{ such that } a_\gamma \neq 0.$$

This is a Henselian valuation with value group Γ and residue field $\mathbf{R}$. Thus $\mathbf{R}((t^\Gamma))$ is a real closed field.

Order $\mathbf{R}((t^\Gamma))$ by

$$f = \sum a_\gamma t^\gamma > 0 \Leftrightarrow a_{v(f)} > 0.$$

Note: ϵ is infinitesimal iff $v(\epsilon) > 0$.

In particular t is infinitesimal and t^{-1} is infinite.

We can define a restricted exponential on $\mathbf{R}((t^\Gamma))$. Suppose $x \in \mathbf{R}((t^\Gamma))$ and $0 \leq x \leq 1$. Then $x = r + \epsilon$ where $r \in \mathbf{R} \cap [0, 1]$ and $v(\epsilon) > 0$. Define

$$\widehat{E}(x) = e^r \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}.$$

This is well defined!

Theorem. $(\mathbf{R}((t^\Gamma)), +, \cdot, <, \widehat{E}) \models Th(\mathbf{R}_{\mathrm{e}})$

Remarks. - The above proof is based on the Denef-van den Dries quantifier elimination for $\mathbf{R}_{\mathrm{an}}$.

-$Th(\mathbf{R}_{\mathrm{e}})$ is model complete.

-(Kuhlman-Kuhlman-Shelah) $\mathbf{R}((t^\Gamma))$ can never support a full exponential.

Let $\Gamma_0 = \mathbf{R}$, $K_0 = \mathbf{R}((t^{\mathbf{R}}))$ and $O_0 = \{x \in K_0 : v(x) \geq 0\}$.

Define $\widehat{E_0} : O_0 \to K_0$ by

$$\widehat{E_0}(r + \epsilon) = e^r \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}$$

where $r \in \mathbf{R}$ and $v(\epsilon) > 0$. Since $t^{-1}, t^{-2}, \ldots$ is cofinal in K_0 and we want $E(t^{-1}) > t^{-n}$ for all n, we can not define a define a full exponential on K_0.

Let $J_0 = \{x \in K_0 : \text{supp } x < 0\} \cup \{0\}$. Then $K_0 = J_0 \oplus O_0$.

Let $\Gamma_1 = J_0 \oplus \Gamma \subset K_0$. Let $K_1 = \mathbf{R}((t^{\Gamma_1}))$. Define $E_0 : K_0 \to K_1$ by

$$E_0(\alpha + \beta) = t^{-\alpha} \widehat{E_0}(\beta)$$

for $\alpha \in J_0$, $\beta \in O_0$.

Note that $E_0(t^{-1}) = t^{-t^{-1}} > t^{-n}$.

There is not enough room in K_1 for $E(E(t^{-1}))$.

Note: $K_1 = K_0((t^{J_0}))$.

We build $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \ldots$ and $K_0 \subset K_1 \subset K_2 \subset \ldots$ such that $K_i = \mathbf{R}((t^{\Gamma_i}))$ and Γ_i is a convex subgroup of Γ_{i+1}.

Let $O_i = \{x \in K_i : v(x) \geq g \text{ for some } g \in \Gamma_{i-1}\}$ and $J_i = \{x \in K_i : \text{supp } x < \Gamma_{i-1}\}$.

If $x \in O_i$, $x = \alpha + +\epsilon$ with $\alpha \in K_{i-1}$ and $v(\epsilon) > \Gamma_{i-1}$, then

$$\widehat{E_i}(x) = E_{i-1}(\alpha) + \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}.$$

Let $\Gamma_{i+1} = J_i \oplus \Gamma_i$.
Let $E_i : K_i \to K_{i+1}$ by

$$E_i(\alpha + \beta) = t^{-\alpha}\widehat{E_i}(\beta)$$

for $\alpha \in J_i$ and $\beta \in O_i$.

Let $\mathbf{R}((t))^{\mathrm{E}} = \bigcup_{n=0}^{\infty} K_n$ and $E = \bigcup_{n=0}^{\infty} E_n$.

Note: $K_{n+1} = K_n((t^{J_n}))$. If $f \in K_n$

$$f = \sum_{\alpha \in J_n} f_\alpha t^\alpha = \sum_{\alpha \in J_n} f_\alpha E(-\alpha)$$

We have ensured that E1)-E4) hold. What about logs?

If $x \in \mathbf{R}((t))^{\mathrm{E}}$ with $x > 0$, then

$$x = at^{\alpha + r}(1 + \epsilon)$$

where $a \in \mathbf{R}$, $a > 0$, supp $\alpha < 0$, $r \in \mathbf{R}$ and $v(\epsilon) > 0$. Then

$$\begin{aligned} \log x &= \log a + \log t^{\alpha} + r \log t + \log(1 + \epsilon) \\ &= \log a - \alpha + r \log t + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\epsilon^n}{n}. \end{aligned}$$

Thus the only obstruction is that there is no logarithm for t in $\mathbf{R}((t))^{\mathrm{E}}$.

Adding Logarithms

Let $L_0 = \mathbf{R}((t))^{\mathrm{E}}$.
Let $t_1 = \frac{1}{\log(t^{-1})}$
Formally build $L_1 = \mathbf{R}((t_1))^{\mathrm{E}}$.

Since $t = E(-1/t_1)$, we can view $L_0 \subset L_1$.
Since $\log t = -1/t_1 \in L_1$ **every element of L_0 has a logarithm in L_1.**

But $L_0 \cong L_1$, so we can build

$$L_0 \subset L_1 \subset \dots$$

such that $(L_0, L_1) \cong (L_n, L_{n+1})$.

Let $\mathbf{R}((t))^{\mathrm{LE}} = \bigcup L_i$.

Theorem. $\mathbf{R}((t))^{\mathrm{LE}} \models Th(\mathbf{R}_{\exp})$

We give a simple application.

Let $e_0(x) = x$ and $e_{n+1}(x) = e^{e_n(x)}$.

Corollary. Suppose $f : \mathbf{R} \to \mathbf{R}$ is definable in $\mathbf{R}_{\exp}$. Then there are $m, n \in \mathbf{N}$ such that $f(x) < e_n(x)$ for all $x > m$.

Proof.

If not then, by compactness there is a nonstandard model $\mathcal{M}$ with $a \in \mathcal{M}$, $a > \mathbf{R}$ and $f(a) > e_n(a)$ for all $n \in \mathbf{N}$.

By o-minimality if this is true for some infinite element in some model it is true for every infinite element in every model. Thus $f(t^{-1}) > e_n(t^{-1})$ for all $n \in \mathbf{N}$.

But it is easy to see that

$$E(t^{-1}), E(E(t^{-1})), E(E(E(t^{-1}))) \ldots$$

is cofinal in $\mathbf{R}((t))^{\mathrm{LE}}$. Thus $f(t^{-1}) < e_n(t^{-1})$ for some n, a contradiction.

Hardy's Problem

Let $i(x)$ be the inverse to $x \mapsto x \log x$. Then $i(x) \sim x/\log x$. Let

$$i(x) = \frac{x}{\log(x)}(1+\epsilon(x))$$

where $\epsilon(x) \to 0$ as $x \to +\infty$. Then

$$\begin{aligned} x &= i(x)\log i(x) \\ &= \frac{x}{\log x}(1+\epsilon(x))(\log x - \log\log x \\ &\qquad\qquad + \log(1+\epsilon(x))). \end{aligned}$$

Let

$$F(u,v,\epsilon) = (1+\epsilon)(1-u+v\log(1+\epsilon)) - 1.$$

Then

$$F\Big(\frac{\log\log x}{\log x}, \frac{1}{\log x}, \epsilon(x)\Big) = 0.$$

Since F is analytic at (0,0,0), $F(0,0,0) = 0$ and $\frac{\partial F}{\partial \epsilon}(0,0,0) = 1$, there is, by the implicit function theorem, an analytic function f defined on an open $U \subset \mathbf{R}^2$ such that $(0,0) \in U$, $f(0,0) = 0$ and

$$F(u,v,f(u,v)) = 0.$$

Clearly $\epsilon(x) = f(\frac{\log\log x}{\log x}, \frac{1}{\log x})$.

Suppose $g(x)$ is an LE-function such that $e^{i(x)} \sim g(x)$. Then

$$\lim_{x \to +\infty} \frac{e^{i(x)}}{g(x)} = 1$$

and

$$\lim_{x \to +\infty} (i(x) - \log g(x)) = 0.$$

Thus $i(t^{-1}) - \log(g(t^{-1}))$ is infinitesimal in $\mathbf{R}((t))^{\mathrm{LE}}$. That is $v(i(t^{-1}) - \log(g(t^{-1})) > 0$. But

$$i(t^{-1}) = \frac{t^{-1}}{\log t^{-1}} \Big(1 + f\Big(\frac{\log\log x}{\log x}, \frac{1}{\log t^{-1}}\Big)\Big).$$

Since f is analytic, $f(x, y) = \sum a_{i,j} x^i y^j$.

A straightforward calculation shows supp $i(t^{-1}) < 0$.

Since $v(i(t^{-1}) - \log(g(t^{-1}))) > 0$, $i(t^{-1})$ is an initial segment of $\log g(t^{-1})$.

Lemma. If f is a definable function, h is an LE function and $f(t^{-1}) = h(t^{-1})$, then $f(x)$ is agrees with an LE-function for sufficiently large x.

But $i(x)$ is not an LE-function, a contradiction.

Differential Equations in $\mathbf{R}((t))^{\mathrm{LE}}$

We first define derivation D on $\mathbf{R}((t))^{\mathrm{LE}}$ (which we think of a $\frac{d}{dt}$).

On K_0:

$$D\Big(\sum_{r\in\mathbf{R}} a_r t^r\Big) = \sum r a_r t^{r-1}.$$

Let $f \in K_{n+1}$, say $f = \sum_{\alpha\in J_n} f_\alpha E(-\alpha)$. Then

$$D(f) = \sum_{\alpha\in J_n} (D(f_\alpha) - f_\alpha D(\alpha))t^\alpha.$$

Use the isomorphism between $\mathbf{R}((t))^{\mathrm{E}}$ and each L_n to extend D to $\mathbf{R}((t))^{\mathrm{LE}}$.

Let $x = t^{-1}$. We define a new derivation by $f' = -t^2 D(f)$ which corresponds to $\frac{d}{dx}$. Note $\frac{dt}{dx} = -t^2$.

Lemma. i) $f' = 0$ if and only if $f \in \mathbf{R}$.

ii) $E(f)' = E(f)f'$ for all $f \in \mathbf{R}((t))^{\mathrm{LE}}$.

Theorem. If $f \in \mathbf{R}((t))^{\mathrm{LE}}$, there is $g \in \mathbf{R}((t))^{\mathrm{LE}}$ with $g' = f$.

Corollary. If $f, g \in \mathbf{R}(\!(t)\!)^{\mathrm{LE}}$, there is $y \in \mathbf{R}(\!(t)\!)^{\mathrm{LE}}$ such that $y' + fy = g$.

Proof.

Let $h_0, h_1 \in \mathbf{R}(\!(t)\!)^{\mathrm{LE}}$ such that $h_0' = f$ and $h_1' = E(h_0)g$. Then $y = \frac{h_1}{E(h_0)}$ is the desired solution.

Lemma. If $f \in \mathbf{R}(\!(t)\!)^{\mathrm{LE}}$ and $v(f) \geq 0$, then $v(f') > 0$.

Corollary. The differential equation $y'' + y = 0$ has no nontrivial solutions in $\mathbf{R}(\!(t)\!)^{\mathrm{LE}}$.

Proof. Suppose f is a nontrivial solution. Consider $g = (f')^2 + f^2$. Then

$$g' = 2f'f'' + 2ff' = 0.$$

So $g \in \mathbf{R}$.

If $v(f) < 0$, then $|f|$ is infinite and thus g is infinite.

If $v(f) > 0$, then, $v(f') > 0$ and g is infinitesimal.

If $v(f) = 0$, then, $v(f'') > 0$ and $v(f'' + f) = 0$. In any case we have a contradiction.

Definition: A *Hardy field* is a differential field of germs of real functions at $+\infty$.

If $f \neq 0$ is in a Hardy field, then $f(x) \neq 0$ for all sufficiently large x.

Naive Conjecture: There is an intimate relationship between solving differential equations in $\mathbf{R}(\!(t)\!)^{\mathrm{LE}}$ and solving differential equations in Hardy fields extending $\mathbf{R}(x)$.

Theorem. If $f \in L$ then the equation $y'' + fy = 0$ has a nontrivial solution in $\mathbf{R}(\!(t)\!)^{\mathrm{LE}}$ if and only if $f(x) < \frac{1}{4x^2} + \frac{1}{4x^2(\log x)^2} + \frac{1}{4x^2(\log x)^2(\log\log x)^2} + \cdots$.

This looks ugly but it is the desired result. Consider $y'' + \frac{k}{x^2}y = 0$. Guess that $y = x^\lambda$. Then $\lambda^2 - \lambda + k = 0$. If $k \leq 1/4$, $\lambda \in \mathbf{R}$ and x^λ is a solution in a Hardy field. If $k > 1/4$, $\lambda = \mu \pm \nu i$ and the general solutions are of the form $x^\mu(\cos\log(x^{\pm\nu}))$.

By the Sturm comparision theorem if $f(x) < 1/4x^2$, then $y'' + fy = 0$ has a nontrivial solution in a Hardy field, while if $f(x) > k/x^2$ for some $k > 1/4$, then all nontrivial solutions are oscillating.

Consider differential equations $y' = p(x, y)$ for $p \in \mathbf{R}[X, Y]$.

Theorem. 1) $y' = p(x, y)$ has a solution in $\mathbf{R}((t))^{\mathrm{LE}}$ with $v(y') < \mathbf{R}$ if and only if there is a solution f in a Hardy field with $|f(x)| > x^n$ for all large enough x.

2) $y' = p(x, y)$ has a solution in $\mathbf{R}((t))^{\mathrm{LE}}$ with $v(y') > \mathbf{R}$ if and only if there is a solution f in a Hardy field with $|f(x)| < 1/x^n$ for all large enough x.