Metamathematics I
Fall 2001

Problem Set II
Due Monday September 17

1) Let ϕ be an $\mathcal{L}$-sentence and let $\mathcal{A}$ be an $\mathcal{L}$-structure. Prove that if there is an assignment α such that $(\mathcal{A}, \alpha) \models \phi$, then $(\mathcal{A}, \beta) \models \phi$ for all assignments β. [**Hint**: You will probably want to prove this by induction on formulas. You will need to first formulate the right generalization to formulas.]

2)a) Let $\mathcal{L}$ be the language $\{+, 0\}$ and consider the structure $\mathcal{R}$ with universe $\mathbf{R}$ (the real numbers) where $+$ is interpreted as the usual addition and 0 as zero. Show that there is no formula $\phi(v, w)$ such that $\mathcal{R} \models \phi(a, b)$ if and only if $a < b$ for all $a, b \in \mathbf{R}$. (**Hint:** Find a and b and an automorphism of $\mathbf{R}$ such that $a < b$ but $F(a) > F(b)$).

b)† Let $\mathcal{L}$ be the language of rings $\{+, \cdot, 0, 1\}$ and let $\mathcal{R}$ be $\mathbf{R}$ with the usual interpretation. Find a formula $\phi(v, w)$ such that $a < b$ iff $\mathcal{R} \models \phi(a, b)$.

3) Let T be an $\mathcal{L}$-theory. We say that T' is an *axiomatization* of T if $\mathcal{A} \models T$ if and only if $\mathcal{A} \models T'$ for any $\mathcal{L}$-structure $\mathcal{A}$. Suppose T' is an axiomatization of T. Show that $T \models \phi$ if and only if $T' \models \phi$ for all $\mathcal{L}$-sentences ϕ.

4) If ϕ is a sentence, the *spectrum* of ϕ is the set of all natural numbers n such that there is a model of ϕ with exactly n elements.

i) Let $\mathcal{L} = \{E\}$ where E is a binary relation. Write down a sentence ϕ asserting that E is an equivalence relation and every equivalence class has exactly three elements. Show that the spectrum of ϕ is $\{n > 0\colon\ 3 \text{ divides } n\}$.

ii) Let $\mathcal{L} = \{P, Q, f\}$ where P and Q are unary predicates and f is a binary function. Let ϕ be the conjuction of:

$$\exists x \exists y\ x \neq y \wedge P(x) \wedge P(y)$$
$$\exists x \exists y\ x \neq y \wedge Q(x) \wedge Q(y)$$
$$\forall z \exists x \exists y\ P(x) \wedge Q(y) \wedge f(x, y) = z$$
$$\forall x_1 \forall x_2 \forall y_1 \forall y_2\ [(P(x_1) \wedge P(x_2) \wedge Q(y_1) \wedge Q(y_2) \wedge f(x_1, y_1) = f(x_2, y_2)) \rightarrow (x_1 = x_2 \wedge y_1 = y_2)]$$

Show that the spectrum of $\phi = \{n > 3 : n \text{ is not prime}\}$.[1]

iii) Find a sentence with the spectrum $\{n > 0 : n \text{ is a square }\}$.

iv) Find a sentence with the specturm $\{p^n : p \text{ prime } n > 0\}$.

v)** Find a sentence with spectrum $\{p : p \text{ is prime}\}$.

Remarks • An interesting theorem from computer science says that $X \subseteq \mathbf{N}$ is the spectrum of a sentence if and only if X is recognizable in non-deterministic exponential time.

• Open Question: Suppose X is the spectrum of a sentence. Is $\mathbf{N}^+ \setminus X$ also a spectrum?]

[1] **Note**: Another sentence with the same spectrum is the sentence in the language of rings asserting that we have an integral domain *with* zero divisors