Metamathematics I
Problem Set V

Due Wednesday October 24

1) Let $\mathcal{L} = \{+, 0\}$ and let Γ be the $\mathcal{L}$-theory axiomatized by i) axioms for abelian groups, ii) the assertion that every element has order two. Models of Γ can be thought of as vector spaces over the two element field.

a) Show that for any infinite cardinal κ, Γ is κ-categorical.

b) Show Γ is not complete.

c) How can we extend Γ to a complete $\mathcal{L}$-theory?

2) Show that $x \leq y$ and $x < y$ are primitive recursive relations.

3) (Definition by cases): Suppose g and h are primitive recursive functions and P is a primitive recursive predcate. Then f is primitive recursive where:

$$f(\bar{x}) = \begin{cases} g(\bar{x}), & \text{if } P(\bar{x}); \\ h(\bar{x}) & \text{otherwise.} \end{cases}$$

4) Show that the following functions are primitive recursive

a) $\max(x, y) =$ maximum of x and y

b) $\mathrm{lcm}(x, y) =$ smallest number divisible by both x and y

OPTIONAL PROBLEMS

5)** Let $f_0(x) = x + 1$ and let $f_{(n+1)}(x) = f_n^{(x)}(x)$, where $f^{(m)}$ is the m^{th} iterate of f (ie. $f^{(0)}(x) = x$ and $f^{(m+1)}(x) = f(f^{(m)}(x))$).

We say $f << g$ if there is an m_0 such that for all $m > m_0$, $f(m) < g(m)$.

Show that for any primitive recursive $g : \mathbf{N} \to \mathbf{N}$ there is an n such that $g << f_n$. [Hint: Prove this by induction on the definition of primitive recursive functions. To do that you will need to do this for functions in several variables as well. The inductive hypothesis should be that if $g : \mathbf{N}^n \to \mathbf{N}$ is primitive recursive then there in an N and an M such that if $\max(x_1, \ldots, x_n) > M$, then $g(x_1, \ldots, x_n) < f_N(\max(x_1, \ldots, x_n)$.]

6)* Define $F(n, m, y)$ as follows:

$F(0, 0, y) = y + 1$

$F(n + 1, 0, y) = F(n, y, y)$

$F(n, m + 1, y) = F(n, 0, F(n, m, y))$.

Let $A(n) = F(n, 0, n)$. Show that $A >> f_i$ for each f_i in problem 5. Thus A and F are not primitive recursion. This shows that the primitve recursive functions are not closed under this type of recursive definition.