Scott Ranks of Counterexamples to Vaught's Conjecture

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Vaught's Conjecture

Conjecture

If T is a first order theory in a countable language, then

$$I(T, \aleph_0) \leq \aleph_0 \text{ or } I(T, \aleph_0) = 2^{\aleph_0}.$$

Let $\phi \in L_{\omega_1,\omega}$. If $I(\phi, \aleph_0) > \aleph_0$, then there is a perfect set of non-isomorphic countable models.

Morley's Theorem

Theorem (Morley)

Let $\phi \in L_{\omega_1,\omega}$. Then $I(\phi, \aleph_0) \leq \aleph_1$ or there is a perfect set of nonisomorphic models.

So, counterexamples to Vaught's Conjecture have exactly $\aleph_1$ non-isomorphic models.

Moreover ϕ is *scattered*, i.e., for any countable fragment Δ there are only countably many Δ-types.

Scott Rank

We define equivalence relations $\sim_\alpha$ on $\mathcal{M}$ as follows:

- $\overline{a} \sim_0 \overline{b}$ if for any atomic formula ϕ

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{M} \models \phi(\overline{b}) :$$

- for α a limit ordinal, $\overline{a} \sim_\alpha \overline{b}$ if and only if $\overline{a} \sim_\beta \overline{b}$ for all $\beta < \alpha$;
- $\overline{a} \sim_{\alpha+1} \overline{b}$ if and only if $\forall c \exists d \;\; \overline{a}, c \sim_\alpha \overline{b}, d$ and $\forall d \exists c \;\; \overline{a}, c \sim_\alpha \overline{b}, d$.

Scott rank and α-homogenity

We say $\mathcal{M}$ is *α-homogeneous* if

$$\overline{a} \sim_\alpha \overline{b} \Rightarrow \overline{a} \sim_\beta \overline{b} \text{ for all } \beta.$$

For any $\mathcal{M}$ there is $\alpha < |\mathcal{M}|^+$ such that $\mathcal{M}$ is α-homogeneous.

The least such α is the *Scott rank* of $\mathcal{M}$..

For any $\alpha < \omega_1$ a counterexample to Vaught's Conjecture will have at most countably countable many models of Scott rank below α.

Thus a counterexample, must have models of arbitrarily large countable Scott rank.

We will revisit this fact later.

Uncountable Models of First Order Counterexamples

John Baldwin observed:

Theorem

If T is a first order theory where Vaught's Conjecture fails, then $I(T, \aleph_1) = 2^{\aleph_1}$.

Proof

- (Shelah) Vaught's Conjecture holds for ω-stable theories.
- (Shelah) If T is not ω-stable, then $I(T, \aleph_1) = 2^{\aleph_1}$.

What about $L_{\omega_1,\omega}$?

Theorem (Harnik-Makkai)

Suppose $\phi \in L_{\omega_1,\omega}$ is a counterexample to Vaught's Conjecture.

- *There is a model of size $\aleph_1$ that is $L_{\infty,\omega}$-equivalent to a countable model. (In fact there are $\aleph_1$ countable models which are $L_{\infty,\omega}$-equivalent to an uncountable model.)*
- *There is a model of size $\aleph_1$ that is not $L_{\infty,\omega}$-equivalent to a countable model.*

What about $\aleph_2$?

Theorem (Hjorth)

If ϕ is a counterexample to Vaught's Conjecture, then there is a counterexample ψ such that $\psi \models \phi$ and ψ has no models of size $\aleph_2$.

Hjorth's proof uses heavily the descriptive set theory of actions of S_∞ and the construction of ψ from ϕ is a bit mysterious.

Sacks has tried to prove Vaught's Conjecture by showing that counterexamples must have models of size $\aleph_2$.

Harrington's Theorem

Theorem (Harrington)

If ϕ is a counterexample to Vaught's Conjecture, then for all $\alpha < \omega_2$, ϕ has a model of Scott rank at least α.

In particular, $I(\phi, \aleph_1) \geq \aleph_2$.

Question

Can we improve this to $I(\phi, \aleph_1) = 2^{\aleph_1}$?

For the remainder of the talk, I will give a sketch of Harrington's proof.

Ingredient: The Model Existence Game

Let $\phi \in L_{\infty,\omega}$. We define a game G_ϕ.
Let Δ be the smallest fragment containing ϕ and let C be a countable set of new constants.

A play of the G_ϕ looks like:

$$\begin{array}{cc} \text{I} & \text{II} \\ \phi_0 & \\ & s_0 \\ \phi_1 & \\ \vdots & s_1 \\ \vdots & \vdots \end{array}$$

where ϕ_i is $\Delta(C)$ and s_i is a finite set of $\Delta(C)$-sentences.

Player II wins G_ϕ if $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots$ and

- either $\phi_i \in s_i$ or $\neg\phi_i \in s_i$, i.e., s_i commits to ϕ_i or $\neg\phi_i$;
- if $\phi_i = \phi$, then $\phi \in s_i$;
- if $\phi_i = \bigvee \psi_j$ and $\phi_i \in s_i$ then some $\psi_j \in s_i$;
- if $\phi_i = \exists v \psi(v)$ and $\phi_i \in s_i$, then $\psi(c) \in s_i$ for some $c \in C$;
- $c \neq c$ is not in any s_i;
- if $c = d$ is in s_i, then $d \neq c \notin s_i$;
- if $\phi(c), c = d \in s_i$, then $\neg\phi(d) \notin s_i$;
- if $\bigwedge \psi_i \in s_i$, then $\neg\psi_i \notin s_i$ (in particular $\psi, \neg\psi \notin s_i$);
- if $\forall v \phi(v) \in s_i$, then $\neg\phi(c) \notin s_i$.

Observations

- G_ϕ is an closed game –if Player I wins a play of the game there is a stage where it is determined that Player I has won. Thus one of the players has a winning strategy–If Player I doesn't have a winning strategy , Player II wins by avoiding losing positions.

- If there is $\mathcal{M} \models \phi$, then Player II has a winning strategy in G_ϕ– Player II just answers what's true in $\mathcal{M}$, where we assign constants dynamically in a reasonable way.

Observations

- If $\phi \in L_{\omega_1,\omega}$ and Player II has a winning strategy, then there is $\mathcal{M} \models \phi$ –Since $\Delta(C)$ is countable, we can consider a play of the game where Player I plays every $\Delta(C)$-sentence. Then $\bigcup s_i$ is a Henkin set describing a model of ϕ.

- For $\phi \in L_{\omega_1,\omega}$ if Player I has a winning strategy, then there are no models of $\neg\phi$. Thus for $\phi \in L_{\omega_1,\omega}$

$$\models \phi \Leftrightarrow \text{ Player I has winning strategy in } G_{\neg\phi}.$$

$$\phi \text{ is satisfiable } \Leftrightarrow \text{ Player II has a winning strategy in } G_{\phi}$$

For $L_{\infty,\omega}$ things break down

There is an $L_{\omega_2,\omega}$- sentence ϕ in the signature $\{<\}$ such that $\mathcal{M} \models \phi$ if and only if $\mathcal{M}$ is a well-ordering of order type ω_1. Let

$$\psi = \phi \wedge \forall v \bigvee_{i=0}^{\infty} v = c_i.$$

Then ψ has no models.

But Player II has a winning strategy in G_ψ–roughly Player II pretends to play in a generic extension where ω_1 has been collapsed.

Indeed for any $\phi \in L_{\omega_1,\omega}$, Player II has a winning strategy if and only if ϕ has a model in a forcing extension of $\mathbb{V}$.

Define $\models_* \phi$ if and only if Player I has a winning strategy in $G_{\neg\phi}$. We think of this as saying ϕ is *formally valid* or *strongly valid*.

Note that if $\models_* \psi$ then $\models \psi$ and the notions are equivalent for $\psi \in L_{\omega_1,\omega}$.

$\models_*$ has many of the simple properties of $\models$.

- If $\models_* \phi$ and $\models_* \phi \to \psi$, then $\models_* \psi$.
- If c does not occur in $\phi(v)$ or ψ and $\models_* \phi(c) \to \psi$, then $\models_* \exists v \phi(v) \to \psi$.

These can be used by manipulating strategies in the games or by using the forcing characterization, but we will shortly give simpler proofs.

Ingredient: Lévy Absoluteness

Recall: $H(\kappa)$ is the sets that hereditarily have cardinality less than κ, and $HC = H(\aleph_1)$ is the set of hereditarily countable sets.

Theorem (Lévy Absoluteness)

If $\kappa < \lambda$, $H(\kappa) \prec_1 H(\lambda) \prec_1 \mathbb{V}$.

Here $\mathcal{M} \prec_1 \mathcal{N}$ if and only if for any Σ_1-formula $\phi(\overline{v})$ in the language of set theory and and $\overline{a} \in \mathcal{M}$,

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{a}).$$

The Absoluteness of $\models_*$

Lemma

$\models_*$ x is Δ_1 on $H(\kappa)$.

Proof.

If $\phi \in H(\kappa)$, then $\phi \in L_{\kappa,\omega}$ and if Δ is the smallest fragment containing ϕ then $\Delta \in H(\kappa)$. A winning strategy will be a function $\sigma \in H(\kappa)$.

Then $\models_* \phi \ \Leftrightarrow$

$\Leftrightarrow$ Player I has a winning strategy in $G_{\neg\phi}$ (Σ_1)

$\Leftrightarrow$ Player II does not have a winning strategy in $G_{\neg\phi}$ (Π_1)

□

Corollary

If $\models_* \phi$ *and* $\models_* \phi \rightarrow \psi$*, then* $\models_* \psi$*.*

Proof.

The statement that for all ϕ and ψ the Corollary holds is a Π_1-sentence Γ.
Since for $L_{\omega_1,\omega}$ the sentence is true for $\models$, $HC \models \Gamma$.
Thus by Lévy Absoluteness it is true in $\mathbb{V}$ and in any $H(\kappa)$. □

Similar proofs work for other useful simple properties of $\models_*$. For example,

Corollary

If $\models_* \phi \rightarrow \theta_i$ *for* $i \in I$*, then* $\models_* \phi \rightarrow \bigwedge_{i\in I} \theta_i$*.*

Ingredient: Scott ranks of countable models revisited

The most difficult part of the proof is a careful analysis of the countable case.

Let ϕ be a counterexample to Vaught's Conjecture.

Lemma

For any $\alpha < \omega_1$ that is at least the quantifier rank of ϕ there is $\mathcal{M} \models \phi$ that is not α-homogeneous with Scott rank at most $\alpha + \omega$.

We can inductively define formulas $S^n_\alpha(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that for any $\mathcal{M}$

$$\mathcal{M} \models S^n_\alpha(\bar{a}, \bar{b}) \Leftrightarrow \bar{a} \sim_\alpha \bar{b}$$

Using these formulas for any α we define σ_α a sentence asserting that $\mathcal{M}$ is not α-homogeneous.

Let Δ be the smallest fragment containing $\phi \wedge \sigma_\alpha$.

Every formula in Δ has quantifier rank less than $\alpha + \omega$.

Since ϕ is scattered, there are only countably many Δ-types for models of $\phi \wedge \sigma_\alpha$.

Thus there is a Δ-atomic model $\mathcal{M}$ of $\phi \wedge \sigma_\alpha$.

$\mathcal{M}$ has Scott rank at most $\alpha + \omega$.

Finding Scott sentences

Let ϕ be a counterexample.

Lemma

Let $\alpha < \omega_1$. Let $\mathbb{A}$ be an admissible set containing ϕ and α. Then $\mathbb{A}$ contains the Scott sentence of a model that is not α-homogeneous.

Admissible sets are transitive models of "enough set theory".

Sketch: Suppose, for simplicity, that ϕ and α are countable in $\mathbb{A}$.

The set of canonical Scott sentences for models of ϕ of Scott rank at most $\alpha + \omega$ is a countable set that is $\Sigma^1_1(\phi, \alpha)$.

By Harrison's Theorem every such Scott sentence is hyperarithmetic in ϕ, α and hence in $\mathbb{A}$.

More work is needed for the general case.

Harrington's Proof

Fix ϕ . We want to show that for $\alpha < \omega_2$ there is $\mathcal{M} \models \phi$ that is not α-homogeneous.

Let Γ be a sentence of set theory asserting:
For all admissible $\mathbb{A}$ with $\phi \in \mathbb{A}$ and for all $\alpha \in \mathbb{A}$ an ordinal, there is $\Psi \in \mathbb{A}$ such that:

- Ψ is formally satisfiable, i.e. $\not\models_* \neg\Psi$;
- $\models_* \Psi \rightarrow (\phi \wedge \sigma_\alpha)$;
- Ψ is formally complete, i.e., for all θ, $\models_* \Psi \rightarrow \theta$ or $\models_* \Psi \rightarrow \neg\theta$;
- (formal atomicity) for all $\theta(\overline{v})$ if $\models_* \Psi \rightarrow \exists\overline{v}\ \theta(\overline{v})$, then there is $\widehat{\theta}(\overline{v}) \in L_{\mathbb{A}}$ such that
  i) $\models_* \widehat{\theta}(\overline{v}) \rightarrow \theta(\overline{v})$;
  ii) $\models_* \Psi \rightarrow \exists\overline{v}\ \widehat{\theta}(\overline{v})$;
  iii) $\widehat{\theta}(\overline{v})$ is complete.

- Γ is Π_1;
- $HC \models \Gamma$; We choose Ψ to be the Scott sentence of a model of Ψ that is not α-homogeneous and use the fact that $\models$ agrees with $\models_*$ for $L_{\omega_1,\omega}$.
- Thus, by Lévy Absoluteness, $H(\aleph_2) \models \Psi$.

Let $\widehat{\alpha} < \omega_2$ and let $\mathbb{A} \in H(\aleph_2)$ be an admissible set containing $\phi, \widehat{\alpha}$. Let Ψ be as in Γ.

Our remaining problem is that $\not\models_* \neg\Psi$ (i.e., formal satisfiability) is not enough to conclude there is a model of Ψ.

Let $C = \{c_\alpha : \alpha < \omega_1\}$ be a new set of constants and let $C_\alpha = \{c_\beta : \beta < \alpha\}$.

Let $\{\exists v\ \delta_\alpha(v) : \alpha < \omega_1\}$ list all $L_{\mathbb{A}}(C)$-formulas such that $\models_* \exists v\ \delta_\alpha(v)$. We assume $\delta_\alpha \in L_{\mathbb{A}}(C_\alpha)$.
For example, we include all formulas

$$\exists v(\exists w \psi(w) \rightarrow \psi(v))$$

which will help us Henkinize.

We build $\Sigma_0 \subset \cdots \subset \Sigma_\alpha \subset \ldots$, $\alpha < \omega_1$ where Σ_α is a countable set of $L_{\mathbb{A}}(C_\alpha)$-sentences such that:

- $\Psi \in \Sigma_0$;
- For all $\overline{c} \in C_\alpha$, there is $\theta(\overline{c}) \in \Sigma_\alpha \cap L_{\mathbb{A}}(\overline{c})$ such that $\theta(\overline{v})$ is complete;
- $\delta_\alpha(c_\alpha) \in \Sigma_{\alpha+1}$;
- If $\theta_1, \theta_2 \in \Sigma_\alpha$, then there is no ψ such that $\models_* \theta_1 \rightarrow \psi$ and $\models_* \theta_2 \rightarrow \neg\psi$.

Let

$$H = \{\psi \in L_{\mathbb{A}}(C) : \models_* \theta \rightarrow \psi \text{ for some } \theta \in \bigcup_{\alpha<\omega_1} \Sigma_\alpha\}.$$

$$H = \{\psi \in L_{\mathbb{A}}(C) : \models_* \theta \rightarrow \psi \text{ for some } \theta \in \bigcup_{\alpha<\omega_1} \Sigma_\alpha\}.$$

For any $\psi(\overline{c}) \in L_{\mathbb{A}}(C)$. Suppose $\overline{c} \in C_\alpha$. There is $\theta(\overline{v})$ complete such that $\theta(\overline{c}) \in \Sigma_\alpha$. Then $\models_* \theta(\overline{v}) \rightarrow \psi(\overline{v})$ or $\models_* \theta(\overline{v}) \rightarrow \neg\psi(\overline{v})$. Thus one of $\psi(\overline{c})$, $\neg\psi(\overline{c})$ is in H.

H is Henkinized.

Suppose $\bigvee \psi_i(\overline{c}) \in H$. We claim that $\psi_i(\overline{c}) \in H$ for some i.
There is $\theta(\overline{c}) \in \bigcup \Sigma_\alpha$ such that $\theta(\overline{v})$ is complete. If $\models_* \theta(\overline{c}) \rightarrow \psi_i(\overline{c})$, then $\psi_i(\overline{c}) \in H$, so assume $\models_* \theta(\overline{c}) \rightarrow \neg\psi_i(\overline{c})$ for all i.
But then $\models_* \theta(\overline{c}) \rightarrow \bigwedge \neg\psi_i(\overline{c})$, a contradiction.

Thus we can build a canonical Henkin model of H.

Constructing the Σ_α

- $\Sigma_0 = \{\Psi\}$.
- Given Σ_α. Let $C_\alpha = \{d_0, d_1, \ldots\}$.

  There is $\psi_n(d_0, \ldots, d_{n-1}) \in \Sigma_\alpha$ complete. We build $\theta_n(d_0, \ldots, d_{n-1}, c_\alpha)$ complete.

  i) Choose $\theta_0(v) \in L_{\mathbb{A}}$ complete such that $\models_* \Psi \to \exists v \theta_0(v)$ and $\models_* \theta_0(v) \to \delta_\alpha(v)$.

  ii) Given θ_n find θ_{n+1} complete such that

$$\models_* \theta_{n+1}(d_0, \ldots, d_n, c_\alpha) \to (\theta_n(d_0, \ldots, d_{n-1}, c_\alpha) \wedge \psi_{n+1}(d_0, \ldots, d_n)).$$