Scott Ranks of Counterexamples to Vaught’s Conjecture

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Conjecture

If $T$ is a first order theory in a countable language, then

\[ I(T, \aleph_0) \leq \aleph_0 \text{ or } I(T, \aleph_0) = 2^{\aleph_0}. \]

Let $\phi \in L_{\omega_1, \omega}$. If $I(\phi, \aleph_0) > \aleph_0$, then there is a perfect set of non-isomorphic countable models.
Morley’s Theorem

**Theorem (Morley)**

Let $\phi \in L_{\omega_1,\omega}$. Then $I(\phi, \aleph_0) \leq \aleph_1$ or there is a perfect set of nonisomorphic models.

So, counterexamples to Vaught’s Conjecture have exactly $\aleph_1$ non-isomorphic models.

Moreover $\phi$ is scattered, i.e., for any countable fragment $\Delta$ there are only countably many $\Delta$-types.
We define equivalence relations $\sim_{\alpha}$ on $\mathcal{M}$ as follows:

- $\bar{a} \sim_0 \bar{b}$ if for any atomic formula $\phi$

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{M} \models \phi(\bar{b}) :$$

- for $\alpha$ a limit ordinal, $\bar{a} \sim_{\alpha} \bar{b}$ if and only if $\bar{a} \sim_{\beta} \bar{b}$ for all $\beta < \alpha$;

- $\bar{a} \sim_{\alpha+1} \bar{b}$ if and only if $\forall c \exists d \quad \bar{a}, c \sim_{\alpha} \bar{b}, d$ and $\forall d \exists c \quad \bar{a}, c \sim_{\alpha} \bar{b}, d$. 
Scott rank and $\alpha$-homogenity

We say $\mathcal{M}$ is $\alpha$-homogeneous if

$$\bar{a} \sim_\alpha \bar{b} \Rightarrow \bar{a} \sim_\beta \bar{b} \text{ for all } \beta.$$ 

For any $\mathcal{M}$ there is $\alpha < |\mathcal{M}|^+$ such that $\mathcal{M}$ is $\alpha$-homogeneous.

The least such $\alpha$ is the Scott rank of $\mathcal{M}$.

For any $\alpha < \omega_1$ a counterexample to Vaught’s Conjecture will have at most countably countable many models of Scott rank below $\alpha$.

Thus a counterexample, must have models of arbitrarily large countable Scott rank.

We will revisit this fact later.
John Baldwin observed:

**Theorem**

If $T$ is a first order theory where Vaught’s Conjecture fails, then $I(T, \aleph_1) = 2^{\aleph_1}$.

**Proof**

- (Shelah) Vaught’s Conjecture holds for $\omega$-stable theories.
- (Shelah) If $T$ is not $\omega$-stable, then $I(T, \aleph_1) = 2^{\aleph_1}$. 
What about $L_{\omega_1,\omega}$?

**Theorem (Harnik-Makkai)**

Suppose $\phi \in L_{\omega_1,\omega}$ is a counterexample to Vaught's Conjecture.

- There is a model of size $\aleph_1$ that is $L_{\infty,\omega}$-equivalent to a countable model. (In fact there are $\aleph_1$ countable models which are $L_{\infty,\omega}$-equivalent to an uncountable model.)
- There is a model of size $\aleph_1$ that is not $L_{\infty,\omega}$-equivalent to a countable model.
What about $\aleph_2$?

**Theorem (Hjorth)**

*If $\phi$ is a counterexample to Vaught’s Conjecture, then there is a counterexample $\psi$ such that $\psi \models \phi$ and $\psi$ has no models of size $\aleph_2$.*

Hjorth’s proof uses heavily the descriptive set theory of actions of $S_\infty$ and the construction of $\psi$ from $\phi$ is a bit mysterious.

Sacks has tried to prove Vaught’s Conjecture by showing that counterexamples must have models of size $\aleph_2$. 
Harrington’s Theorem

**Theorem (Harrington)**

*If* $\phi$ *is a counterexample to Vaught’s Conjecture, then for all* $\alpha < \omega_2$, *$\phi$ has a model of Scott rank at least* $\alpha$.

*In particular,* $I(\phi, \aleph_1) \geq \aleph_2$.

**Question**

*Can we improve this to* $I(\phi, \aleph_1) = 2^{\aleph_1}$? 

For the remainder of the talk, I will give a sketch of Harrington’s proof.
Ingredient: The Model Existence Game

Let $\phi \in L_{\infty, \omega}$. We define a game $G_\phi$.
Let $\Delta$ be the smallest fragment containing $\phi$ and let $C$ be a countable set of new constants.
A play of the $G_\phi$ looks like:

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I          II
\phi_0
\quad s_0
\phi_1
\quad \vdots
\quad s_1
\quad \vdots
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where $\phi_i$ is $\Delta(C)$ and $s_i$ is a finite set of $\Delta(C)$-sentences.
Player II wins $G_{\phi}$ if $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots$ and

- either $\phi_i \in s_i$ or $\neg \phi_i \in s_i$, i.e., $s_i$ commits to $\phi_i$ or $\neg \phi_i$;
- if $\phi_i = \phi$, then $\phi \in s_i$;
- if $\phi_i = \bigvee \psi_j$ and $\phi_i \in s_i$ then some $\psi_j \in s_i$;
- if $\phi_i = \exists v \psi(v)$ and $\phi_i \in s_i$, then $\psi(c) \in s_i$ for some $c \in C$;
- $c \neq c$ is not in any $s_i$;
- if $c = d$ is in $s_i$, then $d \neq c \notin s_i$;
- if $\phi(c), c = d \in s_i$, then $\neg \phi(d) \notin s_i$;
- if $\bigwedge \psi_i \in s_i$, then $\neg \psi_i \notin s_i$ (in particular $\psi, \neg \psi \notin s_i$);
- if $\forall v \phi(v) \in s_i$, then $\neg \phi(c) \notin s_i$. 
Observations

- $G_\phi$ is an closed game –if Player I wins a play of the game there is a stage where it is determined that Player I has won. Thus one of the players has a winning strategy–If Player I doesn’t have a winning strategy , Player II wins by avoiding losing positions.

- If there is $M \models \phi$, then Player II has a winning strategy in $G_\phi$–Player II just answers what’s true in $M$, where we assign constants dynamically in a reasonable way.
Observations

- If $\phi \in L_{\omega_1,\omega}$ and Player II has a winning strategy, then there is $\mathcal{M} \models \phi$ – Since $\Delta(C)$ is countable, we can consider a play of the game where Player I plays every $\Delta(C)$-sentence. Then $\bigcup s_i$ is a Henkin set describing a model of $\phi$.

- For $\phi \in L_{\omega_1,\omega}$ if Player I has a winning strategy, then there are no models of $\neg \phi$. Thus for $\phi \in L_{\omega_1,\omega}$

  $$\models \phi \iff \text{Player I has winning strategy in } G_{\neg \phi}.$$  

  $\phi$ is satisfiable $\iff$ Player II has a winning strategy in $G_{\phi}$
For $L_{\infty, \omega}$ things break down

There is an $L_{\omega_2, \omega}$-sentence $\phi$ in the signature $\{<\}$ such that $\mathcal{M} \models \phi$ if and only if $\mathcal{M}$ is a well-ordering of order type $\omega_1$. Let

$$\psi = \phi \land \forall \nu \bigvee_{i=0}^{\infty} \nu = c_i.$$ 

Then $\psi$ has no models.

But Player II has a winning strategy in $G_\psi$—roughly Player II pretends to play in a generic extension where $\omega_1$ has been collapsed.

Indeed for any $\phi \in L_{\omega_1, \omega}$, Player II has a winning strategy if and only if $\phi$ has a model in a forcing extension of $\mathbb{V}$. 
Define $\models^* \phi$ if and only if Player I has a winning strategy in $G_{\neg \phi}$. We think of this as saying $\phi$ is formally valid or strongly valid.

Note that if $\models^* \psi$ then $\models \psi$ and the notions are equivalent for $\psi \in L_{\omega_1,\omega}$.

$\models^*$ has many of the simple properties of $\models$.

- If $\models^* \phi$ and $\models^* \phi \to \psi$, then $\models^* \psi$.
- If $c$ does not occur in $\phi(v)$ or $\psi$ and $\models^* \phi(c) \to \psi$, then $\models^* \exists v \phi(v) \to \psi$.

These can be used by manipulating strategies in the games or by using the forcing characterization, but we will shortly give simpler proofs.
Recall: $H(\kappa)$ is the sets that hereditarily have cardinality less than $\kappa$, and $HC = H(\aleph_1)$ is the set of hereditarily countable sets.

**Theorem (Lévy Absoluteness)**

If $\kappa < \lambda$, $H(\kappa) \prec_1 H(\lambda) \prec_1 V$.

Here $\mathcal{M} \prec_1 \mathcal{N}$ if and only if for any $\Sigma_1$-formula $\phi(\overline{v})$ in the language of set theory and and $\overline{a} \in \mathcal{M}$,

$$\mathcal{M} \models \phi(\overline{a}) \iff \mathcal{N} \models \phi(\overline{a}).$$
The Absoluteness of $\models^*$

Lemma

$\models^* x$ is $\Delta_1$ on $H(\kappa)$.

Proof.

If $\phi \in H(\kappa)$, then $\phi \in L_{\kappa,\omega}$ and if $\Delta$ is the smallest fragment containing $\phi$ then $\Delta \in H(\kappa)$. A winning strategy will be a function $\sigma \in H(\kappa)$.

Then $\models^* \phi \iff$

$\iff$ Player I has a winning strategy in $G_{\neg \phi} (\Sigma_1)$

$\iff$ Player II does not have a winning strategy in $G_{\neg \phi} (\Pi_1)$
Corollary

If $\models_\ast \phi$ and $\models_\ast \phi \rightarrow \psi$, then $\models_\ast \psi$.

Proof.

The statement that for all $\phi$ and $\psi$ the Corollary holds is a $\Pi_1$-sentence $\Gamma$. Since for $L_{\omega_1,\omega}$ the sentence is true for $\models$, $HC \models \Gamma$. Thus by Lévy Absoluteness it is true in $\mathbb{V}$ and in any $H(\kappa)$.

Similar proofs work for other useful simple properties of $\models_\ast$. For example,

Corollary

If $\models_\ast \phi \rightarrow \theta_i$ for $i \in I$, then $\models_\ast \phi \rightarrow \bigwedge_{i \in I} \theta_i$. 
The most difficult part of the proof is a careful analysis of the countable case.

Let $\phi$ be a counterexample to Vaught’s Conjecture.

**Lemma**

*For any $\alpha < \omega_1$ that is at least the quantifier rank of $\phi$ there is $\mathcal{M} \models \phi$ that is not $\alpha$-homogeneous with Scott rank at most $\alpha + \omega$.*

We can inductively define formulas $S^n_\alpha(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that for any $\mathcal{M}$

$$\mathcal{M} \models S^n_\alpha(\bar{a}, \bar{b}) \iff \bar{a} \sim_\alpha \bar{b}$$

Using these formulas for any $\alpha$ we define $\sigma_\alpha$ a sentence asserting that $\mathcal{M}$ is not $\alpha$-homogeneous.
Let $\Delta$ be the smallest fragment containing $\phi \land \sigma_\alpha$.

Every formula in $\Delta$ has quantifier rank less than $\alpha + \omega$.

Since $\phi$ is scattered, there are only countably many $\Delta$-types for models of $\phi \land \sigma_\alpha$.

Thus there is a $\Delta$-atomic model $\mathcal{M}$ of $\phi \land \sigma_\alpha$.

$\mathcal{M}$ has Scott rank at most $\alpha + \omega$. 
Finding Scott sentences

Let $\phi$ be a counterexample.

**Lemma**

Let $\alpha < \omega_1$. Let $A$ be an admissible set containing $\phi$ and $\alpha$. Then $A$ contains the Scott sentence of a model that is not $\alpha$-homogeneous.

Admissible sets are transitive models of “enough set theory”.

**Sketch:** Suppose, for simplicity, that $\phi$ and $\alpha$ are countable in $A$.

The set of canonical Scott sentences for models of $\phi$ of Scott rank at most $\alpha + \omega$ is a countable set that is $\Sigma^1_1(\phi, \alpha)$.

By Harrison’s Theorem every such Scott sentence is hyperarithmetic in $\phi, \alpha$ and hence in $A$.

More work is needed for the general case.
Fix \( \phi \). We want to show that for \( \alpha < \omega_2 \) there is \( \mathcal{M} \models \phi \) that is not \( \alpha \)-homogeneous.

Let \( \Gamma \) be a sentence of set theory asserting:

For all admissible \( \mathbb{A} \) with \( \phi \in \mathbb{A} \) and for all \( \alpha \in \mathbb{A} \) an ordinal, there is \( \Psi \in \mathbb{A} \) such that:

- \( \Psi \) is formally satisfiable, i.e. \( \not\models_{\mathbb{A}} \neg \Psi \);
- \( \models_{\mathbb{A}} \Psi \rightarrow (\phi \land \sigma_\alpha) \);
- \( \Psi \) is formally complete, i.e., for all \( \theta \), \( \models_{\mathbb{A}} \Psi \rightarrow \theta \) or \( \models_{\mathbb{A}} \Psi \rightarrow \neg \theta \);
- (formal atomicity) for all \( \theta(\vec{v}) \) if \( \models_{\mathbb{A}} \Psi \rightarrow \exists \vec{v} \, \theta(\vec{v}) \), then there is \( \hat{\theta}(\vec{v}) \in L_{\mathbb{A}} \) such that
  - i) \( \models_{\mathbb{A}} \hat{\theta}(\vec{v}) \rightarrow \theta(\vec{v}) \);
  - ii) \( \models_{\mathbb{A}} \Psi \rightarrow \exists \vec{v} \, \hat{\theta}(\vec{v}) \);
  - iii) \( \hat{\theta}(\vec{v}) \) is complete.
• $\Gamma$ is $\Pi_1$;
• $HC \models \Gamma$; We choose $\Psi$ to be the Scott sentence of a model of $\Psi$ that is not $\alpha$-homogeneous and use the fact that $\models$ agrees with $\models^*$ for $L_{\omega_1,\omega}$.
• Thus, by Lévy Absoluteness, $H(\aleph_2) \models \Psi$.

Let $\hat{\alpha} < \omega_2$ and let $A \in H(\aleph_2)$ be an admissible set containing $\phi, \hat{\alpha}$. Let $\Psi$ be as in $\Gamma$.

Our remaining problem is that $\not\models^* \neg\Psi$ (i.e., formal satisfiability) is not enough to conclude there is a model of $\Psi$. 

Let $C = \{ c_\alpha : \alpha < \omega_1 \}$ be a new set of constants and let $C_\alpha = \{ c_\beta : \beta < \alpha \}$.

Let $\{ \exists v \delta_\alpha(v) : \alpha < \omega_1 \}$ list all $L_A(C)$-formulas such that $\models^* \exists v \delta_\alpha(v)$. We assume $\delta_\alpha \in L_A(C_\alpha)$.

For example, we include all formulas

$$\exists v (\exists w \psi(w) \rightarrow \psi(v))$$

which will help us Henkinize.
We build $\Sigma_0 \subset \cdots \subset \Sigma_\alpha \subset \ldots$, $\alpha < \omega_1$ where $\Sigma_\alpha$ is a countable set of $L_\mathbb{A}(C_\alpha)$-sentences such that:

- $\Psi \in \Sigma_0$;
- For all $\bar{c} \in C_\alpha$, there is $\theta(\bar{c}) \in \Sigma_\alpha \cap L_\mathbb{A}(\bar{c})$ such that $\theta(\bar{\nu})$ is complete;
- $\delta_\alpha(c_\alpha) \in \Sigma_{\alpha+1}$;
- If $\theta_1, \theta_2 \in \Sigma_\alpha$, then there is no $\psi$ such that $\models_\ast \theta_1 \rightarrow \psi$ and $\models_\ast \theta_2 \rightarrow \neg \psi$.

Let

$$H = \{ \psi \in L_{\mathbb{A}}(C) : \models_\ast \theta \rightarrow \psi \text{ for some } \theta \in \bigcup_{\alpha < \omega_1} \Sigma_\alpha \}.$$
$H = \{ \psi \in L_\Delta(C) : \models_{\ast} \theta \rightarrow \psi \text{ for some } \theta \in \bigcup_{\alpha < \omega_1} \Sigma_{\alpha} \}.$

For any $\psi(\bar{c}) \in L_\Delta(C)$. Suppose $\bar{c} \in C\alpha$. There is $\theta(\bar{v})$ complete such that $\theta(\bar{c}) \in \Sigma_{\alpha}$. Then $\models_{\ast} \theta(\bar{c}) \rightarrow \psi(\bar{c})$ or $\models_{\ast} \theta(\bar{c}) \rightarrow \neg \psi(\bar{c})$. Thus one of $\psi(\bar{c}), \neg \psi(\bar{c})$ is in $H$.

$H$ is Henkinized.

Suppose $\bigvee \psi_i(\bar{c}) \in H$. We claim that $\psi_i(\bar{c}) \in H$ for some $i$.

There is $\theta(\bar{c}) \in \bigcup \Sigma_{\alpha}$ such that $\theta(\bar{v})$ is complete. If $\models_{\ast} \theta(\bar{c}) \rightarrow \psi_i(\bar{c})$, then $\psi_i(\bar{c}) \in H$, so assume $\models_{\ast} \theta(\bar{c}) \rightarrow \neg \psi_i(\bar{c})$ for all $i$.

But then $\models_{\ast} \theta(\bar{c}) \rightarrow \bigwedge \neg \psi_i(\bar{c})$, a contradiction.

Thus we can build a canonical Henkin model of $H$. 
Constructing the $\Sigma_\alpha$

- $\Sigma_0 = \{\psi\}$.
- Given $\Sigma_\alpha$. Let $C_\alpha = \{d_0, d_1, \ldots\}$. There is $\psi_n(d_0, \ldots, d_{n-1}) \in \Sigma_\alpha$ complete. We build $\theta_n(d_0, \ldots, d_{n-1}, c_\alpha)$ complete.
  i) Choose $\theta_0(v) \in L_\Lambda$ complete such that $\models* \psi \rightarrow \exists v \theta_0(v)$ and $\models* \theta_0(v) \rightarrow \delta_\alpha(v)$.
  ii) Given $\theta_n$ find $\theta_{n+1}$ complete such that $\models* \theta_{n+1}(d_0, \ldots, d_n, c_\alpha) \rightarrow (\theta_n(d_0, \ldots, d_{n-1}, c_\alpha) \land \psi_{n+1}(d_0, \ldots, d_n))$. 