Poker with a Three Card Deck[1]

We start with a three card deck containing one Ace, one King and one Queen. Alice and Bob are each dealt one card at random. There is a pot of $ P (and we assume $P \geq 0$).

Alice may either check or bet $1. If Alice bets, Bob may either call or fold. If Alice checks Bob either checks or bets. If Bob bets, Alice either checks or folds.

We begin with some basic observations.

• **If Alice has a King she should check**: If Alice has a King, Bob has either an Ace or a Queen. If he has an Ace he will call and win. If he has a Queen he will fold. In either case Alice gains nothing by betting.

• **If Alice checks and Bob has a King, he should check** As above, Alice either has an Ace or Queen and if Bob bets Alice will do the right thing and Bob gains nothing.

• **If Alice checks and Bob bets, Alice should fold with a Queen and call with an Ace.**

• **If Alice bets, Bob should call with an Ace and fold with a Queen**

This still leaves several options for Alice and Bob.

- If Alice has an Ace, she has the choice of betting (hoping Bob will call with a King) or checking (hoping Bob will bluff with a Queen).
- l If Alice has a King she will check, but she may either call or fold when Bob bets.
- If Alice has a Queen she has a choice to check or bluff.
- If Bob has a King and Alice bets he has a choice of checking or folding.
- If Bob has a Queen and Alice checks, Bob has a choice of checking or bluffing.

[1]This example is from Bill Chen and Jerrod Ankenman's *The Mathematics of Poker*

There are five interesting probabilities.

$x =$ probability Alice checks with and Ace;
$y =$ probability Alice calls with a King if Bob bets;
$z =$ probability Alice bluffs with a Queen;

$s =$ probability Bob will call with a King when Alice bets;
$t =$ probability that Bob will bluff with a Queen.

We attempt to find what these probabilities are in an optional strategy.

1) x and z should be chosen so that Bob is indifferent between calling and folding when he holds a King and Alice bets.

Suppose Alice bets and Bob holds a King. His expectation if he folds is 0. His expectation if he calls is

$$\frac{x}{2}(-1) + \frac{z}{2}(P+1).$$

Setting this equal to 0, we see that

$$\frac{x}{z} = \frac{1}{P+1} \tag{1}$$

We will try to solve for the other probabilities in terms of x.

2) If Alice checks and Bob holds a Queen, he should bluff so that Alice is indifferent between calling and fold with a King.

Suppose Alice checks a King and Bob bets. Alice's expectation if she folds is 0. Her expectation if she calls is

$$\frac{1}{2}(-1) + \frac{t}{2}(P+1).$$

Thus Alice is indifferent between folding and calling when

$$t = \frac{1}{P+1} \qquad (2).$$

3) If Alice checks a King and Alice should call with a frequency that makes Bob indifferent between checking and bluffing with a Queen.

Suppose Alice checks a King and Bob holds a Queen. Bob's expected value of folding is 0.

If Alice checks, what is the probability she holds an ace?

$$\begin{aligned}
\text{Prob(Alice has an Ace given that she checks)} &= \frac{\text{Prob(Alice has Ace and checks)}}{\text{Prob(Alice checks)}} \\
&= \frac{\frac{1-x}{2}}{\frac{1-x}{2}+\frac{1}{2}} \\
&= \frac{1-x}{2-x}
\end{aligned}$$

and The probability Alice has a King given that she checks is $\frac{1}{2-x}$.

Thus the expected value for Bob for bluffing with a Queen is

$$\frac{1-x}{2-x}(-1)+\frac{y}{2-x}(-1)+\frac{1-y}{2-x}(P)$$

Setting this equal to 0 we see that

$$y = 1+\frac{x-2}{P+1} \tag{3}$$

4) If Alice bets and Bob has a King, he should call to make Alice indifferent between checking a Queen and Bluffing.

If Alice has a Q, the expectation of folding is 0. The expectation of bluffing is

$$\frac{1}{2}(-1)+\frac{s}{2}(-1)+\frac{1-s}{2}P$$

Setting this equal to 0 we see that

$$s = \frac{P-1}{P+1} \tag{4}$$

Summarizing we have:

$$\begin{aligned}
\frac{x}{z} &= \frac{1}{P+1} \\
y &= 1+\frac{x-2}{P+1} \\
s &= \frac{P-1}{P+1}
\end{aligned}$$

$$t = \frac{1}{P+1}$$

We still need to determine x. Alice should choose x to maximize expectation. There are six equally likely possibilities.

case 1 Alice has the Ace, Bob has the King

Alice will bet with probability x. If she bets, Bob will call with probability s. If she checks, Bob will check. Alice's expectation is:

$$x\left(y(P+1)+(1-y)P\right)+(1-x)P.$$

case 2 Alice has the Ace, Bob has the Queen

Alice will bet with probability x. If she bets, Bob will fold. If she checks, Bob will bluff with probability t, in which case Alice will call. Alice's expectation is:

$$xP+(1-x)\left(t(P+1)+(1-t)P\right).$$

case 3 Alice has the King, Bob has the Ace

Alice will check. Bob will bet. Alice will call with probability y. If she calls, she will lose. Alice's expectation is $-y$.

case 4 Alice has the King, Bob has the Queen

Alice will check. Bob will bluff with probability t. If he bets, Alice will call with probability y. Alice's expectation is:

$$ty(P+1)+(1-t)P.$$

case 5 Alice has the Queen, Bob has the Ace

Alice will bluff with probability z. If she bluffs, Bob will call. If Alice checks, Bob bets and Alice folds. Alice's expectation is $-z$.

case 6 Alice has the Queen, Bob has the King

Alice will bluff with probability z. If she bluffs, Bob will call with probability s. If she checks, Bob will call. Alice's expectation is

$$z\left(-y+(1-y)P\right).$$

Each of these cases is equally likely with probability 1/6. Since y, z, s, t are all determined by x. We can find the expectation as a function of x.

With a lot of algebra, we see that

$$E(x)=\left(\left(1-\frac{3}{P+1}\right)x+\frac{3P^2+2P+1}{P+1}\right).$$

As $0 \leq x \leq 1$ and $E(x)$ is linear in x, there are three possibilities:
If $0 \leq P < 2$, then $E(x)$ is maximized when $x = 0$.
If $P = 2$, then $E(x)$ is independent of x.
If $P > 2$, then $E(x)$ is maximized when $x = 1$.

Thus if $P < 2$, Alice's can play $x = 0$ and have expectation

$$\frac{3P^2 + 2P + 1}{P + 1} > \frac{P}{2}$$

, while if $P > 2$ Alice should use $x = 1$ and have expectation

$$\frac{3P^2 + 3P - 1}{P + 1} < \frac{P}{2}.$$

Thus if $P \leq 2$ Alice has an advantage, while if $P > 2$, Bob has an advantage.