Poker with a Three Card Deck

We start with a three card deck containing one Ace, one King and one Queen. Alice and Bob are each dealt one card at random. There is a pot of $P$ (and we assume $P \geq 0$).

Alice may either check or bet $1$. If Alice bets, Bob may either call or fold. If Alice checks Bob either checks or bets. If Bob bets, Alice either checks or folds.

We begin with some basic observations.

• **If Alice has a King she should check**: If Alice has a King, Bob has either an Ace or a Queen. If he has an Ace he will call and win. If he has a Queen he will fold. In either case Alice gains nothing by betting.

• **If Alice checks and Bob has a King, he should check**: As above, Alice either has an Ace or Queen and if Bob bets Alice will do the right thing and Bob gains nothing.

• **If Alice checks and Bob bets, Alice should fold with a Queen and call with an Ace**.

• **If Alice bets, Bob should call with an Ace and fold with a Queen**

This still leaves several options for Alice and Bob.

• If Alice has an Ace, she has the choice of betting (hoping Bob will call with a King) or checking (hoping Bob will bluff with a Queen).

• If Alice has a King she will check, but she may either call or fold when Bob bets.

• If Alice has a Queen she has a choice to check or bluff.

• If Bob has a King and Alice bets he has a choice of checking or folding.

• If Bob has a Queen and Alice checks, Bob has a choice of checking or bluffing.

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1This example is from Bill Chen and Jerrod Ankenman’s *The Mathematics of Poker*
There are five interesting probabilities.

- $x = \text{probability Alice checks with and Ace}$;
- $y = \text{probability Alice calls with a King if Bob bets}$;
- $z = \text{probability Alice bluffs with a Queen}$;
- $s = \text{probability Bob will call with a King when Alice bets}$;
- $t = \text{probability that Bob will bluff with a Queen}$.

We attempt to find what these probabilities are in an optional strategy.

1) $x$ and $z$ should be chosen so that Bob is indifferent between calling and folding when he holds a King and Alice bets.

Suppose Alice bets and Bob holds a King. His expectation if he folds is $0$. His expectation if he calls is

$$\frac{x}{2}(-1) + \frac{z}{2}(P + 1).$$

Setting this equal to $0$, we see that

$$\frac{x}{z} = \frac{1}{P + 1} \quad (1)$$

We will try to solve for the other probabilities in terms of $x$.

2) If Alice checks and Bob holds a Queen, he should bluff so that Alice is indifferent between calling and fold with a King.

Suppose Alice checks a King and Bob bets. Alice’s expectation if she folds is $0$. Her expectation if she calls is

$$\frac{1}{2}(-1) + \frac{t}{2}(P + 1).$$

Thus Alice is indifferent between folding and calling when

$$t = \frac{1}{P + 1} \quad (2).$$

3) If Alice checks a King and Alice should call with a frequency that makes Bob indifferent between checking and bluffing with a Queen.

Suppose Alice checks a King and Bob holds a Queen. Bob's expected value of folding is $0$. 

2
If Alice checks, what is the probability she holds an ace?

\[
\text{Prob(Alice has an Ace given that she checks)} = \frac{\text{Prob(Alice has Ace and checks)}}{\text{Prob(Alice checks)}} = \frac{\frac{1-x}{2}}{\frac{1-x}{2} + \frac{1}{2}} = \frac{1-x}{2-x}
\]

and the probability Alice has a King given that she checks is \(\frac{1}{2-x}\).

Thus the expected value for Bob for bluffing with a Queen is

\[
\frac{1-x}{2-x}(-1) + \frac{y}{2-x}(-1) + \frac{1-y}{2-x}(P)
\]

Setting this equal to 0 we see that

\[
y = 1 + \frac{x - 2}{P + 1} \tag{3}
\]

4) If Alice bets and Bob has a King, he should call to make Alice indifferent between checking a Queen and bluffing.

If Alice has a Q, the expectation of folding is 0. The expectation of bluffing is

\[
\frac{1}{2}(-1) + \frac{s}{2}(-1) + \frac{1-s}{2}P
\]

Setting this equal to 0 we see that

\[
s = \frac{P - 1}{P + 1} \tag{4}
\]

Summarizing we have:

\[
\begin{align*}
x & = \frac{1}{P + 1} \\
z & = \frac{x - 2}{P + 1} \\
y & = 1 + \frac{P - 1}{P + 1} \\
s & = \frac{P - 1}{P + 1}
\end{align*}
\]
\[ t = \frac{1}{P+1} \]

We still need to determine \( x \). Alice should choose \( x \) to maximize expectation. There are six equally likely possibilities.

**case 1** Alice has the Ace, Bob has the King

Alice will bet with probability \( x \). If she bets, Bob will call with probability \( s \). If she checks, Bob will check. Alice’s expectation is:

\[ x(y(P+1) + (1-y)P) + (1-x)P. \]

**case 2** Alice has the Ace, Bob has the Queen

Alice will bet with probability \( x \). If she bets, Bob will fold. If she checks, Bob will bluff with probability \( t \), in which case Alice will call. Alice’s expectation is:

\[ xP + (1-x)(t(P+1) + (1-t)P). \]

**case 3** Alice has the King, Bob has the Ace

Alice will check. Bob will bet. Alice will call with probability \( y \). If she calls, she will lose. Alice’s expectation is \(-y\).

**case 4** Alice has the King, Bob has the Queen

Alice will check. Bob will bluff with probability \( t \). If he bets, Alice will call with probability \( y \). Alice’s expectation is:

\[ ty(P+1) + (1-t)P. \]

**case 5** Alice has the Queen, Bob has the Ace

Alice will bluff with probability \( z \). If she bluffs, Bob will call. If Alice checks, Bob bets and Alice folds. Alice’s expectation is \(-z\).

**case 6** Alice has the Queen, Bob has the King

Alice will bluff with probability \( z \). If she bluffs, Bob will call with probability \( s \). If she checks, Bob will call. Alice’s expectation is:

\[ z(-y + (1-y)P). \]

Each of these cases is equally likely with probability 1/6. Since \( y, z, s, t \) are all determined by \( x \). We can find the expectation as a function of \( x \).

With a lot of algebra, we see that

\[ E(x) = \left( 1 - \frac{3}{P+1} \right) x + \frac{3P^2 + 2P + 1}{P+1} \]
As $0 \leq x \leq 1$ and $E(x)$ is linear in $x$, there are three possibilities:
If $0 \leq P < 2$, then $E(x)$ is maximized when $x = 0$.
If $P = 2$, then $E(x)$ is independent of $x$.
If $P > 2$, then $E(x)$ is maximized when $x = 1$.
Thus if $P < 2$, Alice’s can play $x = 0$ and have expectation
\[
\frac{3P^2 + 2P + 1}{P + 1} > \frac{P}{2}
\]
, while if $P > 2$ Alice should use $x = 1$ and have expectation
\[
\frac{3P^2 + 3P - 1}{P + 1} < \frac{P}{2}
\]
Thus if $P \leq 2$ Alice has an advantage, while if $P > 2$, Bob has an advantage.