A Primer on Infinitary Logic

Fall 2007

These lectures are a brief survey of some elements of the model theory of the infinitary logics $\mathcal{L}_{\infty,\omega}$ and $\mathcal{L}_{\omega_1,\omega}$. It is intended to serve as an introduction to Baldwin's *Categoricity*.

1 Basic Definitions

Let $\mathcal{L}$ be a language. In the logic $\mathcal{L}_{\kappa,\omega}$ we build formulas using the symbols of $\mathcal{L}$, $=$, connectives $\neg$, $\bigwedge$ and $\bigvee$, quantifiers $\forall$ and $\exists$, and variables $\{v_\alpha : \alpha < \kappa\}$.[1]

We define formulas inductively with the usual formation rules for terms, atomic formulas, negation, and quantifiers and the following rule for $\bigwedge$ and $\bigvee$.

If X is a set of $\mathcal{L}_{\kappa,\omega}$ formulas with $|X| < \kappa$, then

$$\bigwedge_{\phi\in X} \phi \text{ and } \bigvee_{\phi\in X} \phi$$

are $\mathcal{L}_{\kappa,\omega}$ formulas.

Satisfaction for $\mathcal{L}_{\kappa,\omega}$-formulas is defined inductively as usual. If $\mathcal{M}$ is an $\mathcal{L}$-structure and σ is an assignment of variables, then

$$\mathcal{M} \models_\sigma \bigwedge_{\phi\in X} \phi \Leftrightarrow \mathcal{M} \models_\sigma \phi \text{ for all } \phi \in X$$

and

$$\mathcal{M} \models_\sigma \bigvee_{\phi\in X} \phi \Leftrightarrow \mathcal{M} \models_\sigma \phi \text{ for some } \phi \in X.$$

For notational symplicity, we use the symbols $\wedge$ and $\vee$ for binary conjunction and disjunctions.

We can inductively define the notions of subformula, free variable, sentence, theory and satisfiability in the usual way.

Exercise 1.1 If ϕ is an $\mathcal{L}_{\kappa,\omega}$-sentence and ψ is a subformula of ϕ then ψ has only finitely many free variables.

We say $\mathcal{M} \equiv_{\kappa,\omega} \mathcal{N}$ if

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$$

[1] For historic reasons we write $\mathcal{L}_{\omega_1,\omega}$ instead of the more notationally consistent $\mathcal{L}_{\aleph_1,\omega}$.

for all $\mathcal{L}_{\kappa,\omega}$ sentences ϕ.

Definition 1.2 Let ϕ is an $\mathcal{L}_{\infty,\omega}$-formula if ϕ is an $\mathcal{L}_{\kappa,\omega}$-formula for some regular cardinal κ.

We define $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ in the obvious way.

Exercise 1.3 Give an example of an set T of $\mathcal{L}_{\omega_1,\omega}$-sentences such that every finite subset is satisfiable but T is not.

Thus the Compactness Theorem fails in infintary languages.[2]

Exercise 1.4 Show that the following classes are $\mathcal{L}_{\omega_1,\omega}$-axiomatizable for appropriate languages $\mathcal{L}$.

i) torsion abelian groups;
ii) finitely generated fields;
iii) linear orders isomorphic to $(\mathbb{Z},<)$;
iv) connected graphs;
v) finite valance graphs;
vi) cycle free graphs.

Exercise 1.5 Show by induction that for each ordinal α there is an $\mathcal{L}_{\infty,\omega}$-sentence Φ_α describing $(\alpha,<)$ up to isomorphism.

Exercise 1.6 Let κ be a cardinal. Show there are $\alpha,\beta < (2^\kappa)^+$ such that $(\alpha,<) \equiv_{\kappa,\omega} (\beta,<)$.

Taken together these two exercises give examples of structures $\mathcal{M},\mathcal{N}$ with $\mathcal{M} \equiv_{\kappa,\omega} \mathcal{N}$ but $\mathcal{M} \not\equiv_{\infty,\omega} \mathcal{N}$ for all κ.[3]

Exercise 1.7 Give an example of a countable language $\mathcal{L}$ and an $\mathcal{L}_{\omega_1,\omega}$-theory T such that every model of T has cardinality at least $2^{\aleph_0}$

Fragments and Downward Löwenheim-Skolem

We will often restrict our attention to subcollections of the set of all $\mathcal{L}_{\kappa,\omega}$-formulas.

For each formula ϕ we define $\sim \phi$, a formal negation of ϕ as follows:

i) for ϕ atomic, $\sim \phi$ is $\neg\phi$;
ii) $\sim (\neg\phi)$ is ϕ;
iii) $\sim \bigwedge_{\phi\in X} \phi$ is $\bigvee_{\phi\in X} \neg\phi$ and $\sim \bigvee_{\phi\in X} \phi$ is $\bigwedge_{\phi\in X} \neg\phi$;
iv) $\sim \exists v\phi$ is $\forall v\neg\phi$ and $\sim \forall v\phi$ is $\exists v\neg\phi$.

[2] By restricting the theories considered, restricting attention to fragments of $\mathcal{L}_{\infty,\omega}$ with strong closure properties and taking a generalized notions of "finite", Barwise proved a infinitary compactness theorem that is useful in some settings. For example, it is useful in descriptive set theory and in studying the constructible hierarchy. As it has not, as of yet, proved very useful in model theory, we will not discuss it here. See [1] or [5] for further discussion.

[3] For a specific example. Suppose $\kappa < \lambda$ are uncountable cardinals. Then $\kappa \equiv_{\omega_1,\omega} \lambda$, but $\kappa \not\equiv_{\infty,\omega} \lambda$. See [6] for details.

Definition 1.8 We say that a set of $\mathcal{L}_{\kappa,\omega}$-formulas $\mathcal{F}$ is a *fragment* if there is an infinite set of variables V such that if $\phi \in \mathcal{F}$, then all variables occuring in ϕ are in V and $\mathcal{F}$ satisfies the following closure properties:[4]

i) all atomic formulas using only the constant symbols of $\mathcal{L}$ and variables from V are in $\mathcal{F}$;

ii) $\mathcal{F}$ is closed under subformula;

iii) If $\phi \in \mathcal{F}$, v is free in ϕ, and t is a term where every variable is in V, then the formula obtained by substituting all free occurences of v with t is in $\mathcal{F}$;

iv) $\mathcal{F}$ is closed under $\sim$;

v) $\mathcal{F}$ is closed under $\neg$, $\wedge$, $\vee$, $\exists v$, and $\forall v$ for $v \in V$.

Exercise 1.9 a) Suppose κ is regular (in particular this holds for $\mathcal{L}_{\omega_1,\omega}$). If T is a set of $\mathcal{L}_{\kappa,\omega}$-sentences with $|T| < \kappa$, then there is $\mathcal{F}$ a fragment of $\mathcal{L}_{\kappa,\omega}$ such that $T \subseteq F$ and $|F| < \kappa$.

b) Show that a) may fail if κ is singular.

We write $\mathcal{M} \equiv_{\mathcal{F}} \mathcal{N}$ and $\mathcal{M} \prec_F \mathcal{N}$ for elementary equivalence and elementary submodels with respect to formulas in $\mathcal{F}$.

Theorem 1.10 (Downward Löwenheim-Skolem) *Let $\mathcal{F}$ be a fragment of $\mathcal{L}_{\kappa,\omega}$ such that any formula $\phi \in \mathcal{F}$ has at most finitely many free variables. Let $\mathcal{M}$ be an $\mathcal{L}$-stucture with $X \subseteq M$. There is $\mathcal{N} \prec_{\mathcal{F}} \mathcal{M}$ with $X \subseteq N$ and $|N| = \max(|\mathcal{F}|, |X|)$.*

In particular if $\mathcal{F}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, then every $\mathcal{L}$-structure has a countable $\mathcal{F}$-elementary submodel.

The proof is a simple generalization of the proof in first-order logic. It is outlined in the following Execise.

Exercise 1.11 a) Prove there is $\mathcal{L}^* \supseteq \mathcal{L}$ and $\mathcal{F}^* \supseteq \mathcal{F}$ and $\mathcal{M}^*$ an $\mathcal{L}^*$-expansion of $\mathcal{M}$ such that $|\mathcal{L}^*|, |\mathcal{F}^*| = |\mathcal{F}|$ and for each $\mathcal{F}^*$-formula $\phi(\overline{v}, w)$ with free variables from $v_1, \ldots, v_n, w$, there is an n-ary function symbol f_ϕ such that

$$\mathcal{M} \models \forall \overline{v}\ (\exists w \phi(\overline{v}, w) \rightarrow \phi(\overline{v}, f_\phi(\overline{v}))$$

b) Prove that if $\mathcal{N}$ is an $\mathcal{L}^*$-substructure of $\mathcal{M}^*$, then $\mathcal{N} \prec_{\mathcal{F}^*} \mathcal{M}^*$.

c) Prove that there is an $\mathcal{L}^*$-substructure $\mathcal{N}$ of $\mathcal{M}^*$ with $X \subseteq N$ and $|\mathcal{N}| = \max(|\mathcal{F}^*|, |X|)$.

Exercise 1.12 For $\mathcal{L}$ and $\mathcal{F}$ define the appropriate notion of *built in Skolem functions*. Prove that for any $\mathcal{L}$ and $\mathcal{F}$ and $\mathcal{F}$-theory T there are $\mathcal{L}^* \supseteq \mathcal{L}$, $\mathcal{F}^* \supseteq \mathcal{F}$ and $T^* \supseteq T$ with $|\mathcal{F}^*| = |T^*| = |\mathcal{F}|$ where T^* has built in Skolem functions, and every $\mathcal{M} \models T$ has an expansion $\mathcal{M}^* \models T$.

Exercise 1.13 In first order logic, the Upward Löwenheim-Skolem is an easy consequence of Compactness. In infinitary logics it is generally false. Let $\mathcal{L} = \{U, S, E, c_1, \ldots, c_n, \ldots\}$ and let ϕ be the conjunction of

[4] We will usually assume $v_1, v_2, \ldots, \in V$.

i) $\forall x\ U(x) \leftrightarrow \neg S(x)$,
ii) $\forall x \forall y\ (E(x,y) \rightarrow U(x) \wedge S(y))$;
iii) $c_i \neq c_j$, for $i \neq j$;
iv) $U(c_i)$ for all i,
v) $\forall y \forall z\ ([S(y) \wedge S(z) \wedge\ \forall x\ ((E(x,y) \leftrightarrow E(x,z))] \rightarrow y = z)$
vi) $\forall x (U(x) \rightarrow \bigvee_{i=1}^{\infty} x = c_i$.

Prove that every model of ϕ has size at most $2^{\aleph_0}$.

$\mathcal{L}_{\omega_1,\omega}$ and omitting first order types

For countable $\mathcal{L}_{\omega_1,\omega}$-theories we can find an equivalent class of models of a first order theory omitting a type.

Theorem 1.14 *Let T be a countable set of $\mathcal{L}_{\omega_1,\omega}$-sentences. There is a countable language $\mathcal{L}^*$ a first order theory T^* and a set of types Γ such that: i) if $\mathcal{M} \models T^*$ and omits all types in Γ, then the $\mathcal{L}$-reduct of $\mathcal{M}$ is a model of T;*

ii) every model of T has an $\mathcal{L}^*$-expansion, that is a model of T^* omitting all types in Γ.

Proof We expand $\mathcal{L}$ and $\mathcal{F}$ so that for each formula ϕ in $\mathcal{F}$ with free variables from $v_1, \ldots, v_n$ we have an n-ary relation symbol R_ϕ we add to T^* a sentences

i) if ϕ is atomic, add
$$\forall \overline{v}\ R_\phi \leftrightarrow \phi$$

ii) if ϕ is $\neg\theta$, add
$$\forall \overline{v}\ R_\phi \leftrightarrow \neg R_\theta$$

iii) if ϕ is $\bigwedge_{\theta \in X} \theta$, add
$$\forall \overline{v}\ R_\phi \rightarrow R_\theta$$

for all $\theta \in X$ and let γ_ϕ be the type

$$\{\neg R_\phi\} \cup \{R_\theta : \theta \in X\}$$

iv) if ϕ is $\bigvee_{\theta \in X} \theta$, add
$$\forall \overline{v}\ R_\theta \rightarrow R_\phi$$

for all θ in X and let γ_ϕ be the type

$$\{R_\phi\} \cup \{\neg R_\theta : \theta \in X\}$$

v) if ϕ is $\exists w \theta$, then add

$$\forall \overline{v}\ R_\phi \leftrightarrow \exists w R_\theta.$$

Let Γ be the collection of types γ_ϕ described above.

Exercise 1.15 a) Suppose $\mathcal{M} \models T^*$ and $\mathcal{M}$ omits every type in Γ. Prove that

$$\mathcal{M} \models \forall \overline{v}\ \phi \leftrightarrow R_\phi$$

for all $\phi \in \mathcal{F}^*$.

Conclude that the $\mathcal{L}$-reduct of a model of T^* is a model of T.

b) Prove that every $\mathcal{M} \models T$ has an expansion that is a model of T^* omitting all types in Γ by interpreting R_ϕ as ϕ.

$\mathcal{L}_{\kappa,\lambda}$

For completeness, we mention the the logics $\mathcal{L}_{\kappa,\lambda}$. In these logics we allow a more general quantification rule.

Suppose ϕ is a formula and $\vec{v}$ is a sequence of variables freely occuring in ϕ with $|\vec{v}| < \lambda$, then

$$\exists \vec{v} \phi$$

is a formula.

Thus we are allowed existential and, by taking negations in the usual way, universal quantification over sequences of variables of cardinality less than λ.

Exercise 1.16 Show there is $\phi \in \mathcal{L}_{\omega_1,\omega_1}$ such that $\mathcal{M} \models \phi$ if and only if $|M| > \aleph_0$.

Exercise 1.17 Show there is an $\mathcal{L}_{\omega_1,\omega_1}$-sentence ϕ such that $(A, <) \models \phi$ if and only if $(A, <)$ is a well-order.

Exercise 1.18 Show that is an $\mathcal{L}_{\omega_1,\omega_1}$-sentence ϕ such that $(A, <) \models \phi$ if and only if $(A, <) \cong (\mathbb{R}, <)$.

Exercise 1.19 Recall that a linear ordering $(A, <)$ is $\aleph_1$-*like* if $|A| = \aleph_1$ and $|\{b : b < a\}| \leq \aleph_0$ for all $a \in A$. Show that the class of $\aleph_1$-like linear orders is $\mathcal{L}_{\omega_1,\omega_1}$-axiomatizable.

2 Back and Forth

We begin with Karp's "algebraic" characterization of $\equiv_{\infty,\omega}$.

Definition 2.1 Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. A *partial isomorphism system* between $\mathcal{M}$ and $\mathcal{N}$ is a collection P of partial $\mathcal{L}$-embeddings $f : A \to N$ where $A \subseteq M$ such that:

i) for all $f \in P$ and $a \in M$ there is $g \in P$ such that $g \supseteq f$ and $a \in \mathrm{dom}(g)$;

ii) for all $g \in P$ and $b \in N$ there is $g \in P$ such that $g \supseteq f$ and $b \in \mathrm{img}(g)$.

We write $\mathcal{M} \cong_p \mathcal{N}$ if there is a partial isomorphism system between $\mathcal{M}$ and $\mathcal{N}$.

Exercise 2.2 If $\mathcal{M}$ and $\mathcal{N}$ are countable and $\mathcal{M} \cong_p \mathcal{N}$, then $\mathcal{M} \cong \mathcal{N}$.

Definition 2.3 Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. The game $G(\mathcal{M}, \mathcal{N})$ is played as follows. At stage n, Player I plays $a_n \in M$ or $b_n \in N$. In the first case Player II responds with $b_n \in N$ and in the later case Player II responds with $a_n \in M$. Player II wins the play of the game is the map $a_n \mapsto b_n$ is an $\mathcal{L}$-embedding.

Theorem 2.4 *The following are equivalent:*

i) $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$*;*

ii) $\mathcal{M} \cong_p \mathcal{N}$*;*

iii) There is partial isomorphism system between $\mathcal{M}$ *and* $\mathcal{N}$ *where every every* $p \in P$ *has finite domain;*

iv) Player II has a winning strategy in $G(\mathcal{M}, \mathcal{N})$*.*

Proof

iii) $\Rightarrow$ ii) is clear.

ii) $\Rightarrow$ i) Let P be a system of partial isomorphisms. We prove that for all $\phi(v_1, \ldots, v_n) \in \mathcal{L}_{\infty,\omega}$, $f \in P$ and $a_1, \ldots, a_n \in \mathrm{dom}(f)$, then

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(f(\overline{a})).$$

We prove this by inducion on formulas. This is clear for atomic formulas and the induction step is obvious for $\neg, \bigwedge$ and $\bigvee$. Suppose $\phi(\overline{v})$ is $\exists w \psi(\overline{v}, w)$.

Suppose $\mathcal{M} \models \exists w \psi(\overline{a}, c)$. There is $g \in P$ with $g \supseteq f$ and $c \in \mathrm{dom}(P)$. By induction, $\mathcal{N} \models \phi(f(\overline(a)), f(c))$, so $\mathcal{N} \models \phi(f(\overline{a}))$.

On the other hand, if $\mathcal{N} \models \psi(f(\overline{a}), d)$. There is $g \in P$ with $g \supseteq f$ and $c \in \mathrm{dom}(g)$ such that $g(c) = d$. By induction, $\mathcal{M} \models \psi(\overline{a}, c)$. Thus $\mathcal{M} \models \exists \phi(\overline{a})$.

This proves the claim. In particular, if ϕ is a sentence $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$.

iv)$\Rightarrow$ iii) Let τ be a winning strategy for Player II. Let P be the set of all maps $f(a_i) = b_i$ where $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are the results of some play of the game where at each stage Player I has played either a_n or b_n and Player II has responded using τ. Since τ is a winning strategy for Player II, each such f is a partial $\mathcal{L}$-embedding. Since Player I can at any stage play any element from M or N, P satisfies i) and ii) in the definition of a partial isomorphism system.

i) $\Rightarrow$ iv) We need one fact.

Claim Suppose $(\mathcal{M}, \overline{a}) \equiv_{\infty,\omega} (\mathcal{N}, \overline{b})$ and $c \in M$, there is $d \in N$ such that

$$(\mathcal{M}, \overline{a}, c) \equiv_{\infty,\omega} (\mathcal{N}, \overline{b}, d).$$

Suppose not. Then for all $d \in \mathcal{N}$ there is ϕ_d such that $\mathcal{M} \models \phi_d((\overline{a}, c)$ and $\mathcal{N} \models \neg\phi_d(\overline{b}, d)$. But that

$$\mathcal{M} \models \exists v \bigwedge_{d \in N} \phi_d(\overline{a}, v)$$

and

$$\mathcal{N} \not\models \exists v \bigwedge_{d \in N} \phi_d(\overline{b}, v)$$

a contradicion.

We now describe Player II's strategy. Player II always has a play to ensure $(\mathcal{M}, a_1, \ldots, a_n) \equiv_{\infty,\omega} (\mathcal{N}, b_1, \ldots, b_n)$. As long as Player II does this the resulting map will be a partial $\mathcal{L}$-embedding.

Exercise 2.5 † Prove that $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ if and only if there is a forcing extension $\mathbb{V}[G]$ of the universe where $\mathcal{M} \cong \mathcal{N}$.

Exercise 2.6 We say that $\mathcal{M} \cong_p^\alpha \mathcal{N}$ if and only if there is a system

$$P_0 \supseteq P_1 \supseteq \ldots \supseteq P_\alpha$$

if and only if for $\beta + 1 \leq \alpha$

i) for any $f \in P_{\beta+1}$ and $a \in M$ there is $g \in P_\beta$ with $g \supseteq f$ and $a \in \text{dom}(g)$;

ii) for any $f \in P_{\beta+1}$ and $b \in N$ there is $g \in P_\beta$ with $g \supseteq f$ and $a \in \text{img}(g)$.

Prove that $\mathcal{M} \cong_{p^\alpha} \mathcal{N}$ if and only if $\mathcal{M} \models \phi \leftrightarrow \mathcal{N} \models \phi$ for all ϕ with quantifer rank less at most α.

Exercise 2.7 We say that $\mathcal{M} \cong_p^\lambda \mathcal{N}$ if there is a system of partial $\mathcal{L}$-embedings P such that if $f \in P$ and $C \subseteq M$ with $|C| < \lambda$ then there is $g \supseteq f$ with $g \in P$ and $C \subseteq \text{dom}(g)$ and if $D \subseteq N$ and $|D| < \lambda$ there is $h \supseteq f$ with $h \in P$ and $D \subseteq \text{img}(h)$.

Prove that $\mathcal{M} \cong_p^\lambda \mathcal{N}$ if and only if $\mathcal{M} \equiv_{\infty,\lambda} \mathcal{N}$.

Exercise 2.8 Prove that any two $\aleph_1$-like DLO are $\mathcal{L}_{\infty,\omega_1}$-equivalent. Hint: Let $(A,<)$ and $(B,<)$ be $\aleph_1$-like orders. We say $A_0 \lhd A$ if A_0 is a countable dense initial segment of A with no top element and no least upper bound in $A \setminus A_0$. Let

$$P = \{f : A_0 \to B_0 : f \text{ an isomorphism }, A_0 \lhd A \text{ and } B_0 \lhd B\}.$$

Note these arguments do not show that A and B are isomorphic. Indeed there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-like DLO. See for example [8] Exercise 5.5.10.

Scott's Theorem

We will show that for any $\mathcal{M}$ there is a single sentence that characterizes $\mathcal{M}$ up to $\equiv_{\infty,\omega}$. Note that for countable models this means that $\mathcal{M}$ is characterized up to elementary equivlance.

For each ordinal α, we will have a relation $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ where $\overline{a} \in M^n$ and $\overline{b} \in N^n$ and $n = 0, 1, 2, \ldots$.

$(\mathcal{M}, \overline{a}) \sim_0 (\mathcal{N}, \overline{b})$ if $\mathcal{M} \models \phi(\overline{a})$ if and only if $\mathcal{N} \models \phi(\overline{b})$ for all atomic $\mathcal{L}$-formulas ϕ.

For all ordinals α, $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{b})$ if for all $c \in M$ there is $d \in N$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$ and for all $d \in N$ there is $c \in M$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$.

For all limit ordinals β, $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{N}, \overline{b})$ if and only if $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ for all $\alpha < \beta$.

If $\mathcal{L}$ is any first-order language and $\mathcal{M}$ is an $\mathcal{L}$-structure we define a sequence of $\mathcal{L}_{\infty,\omega}$-formulas $\phi_{\overline{a},\alpha}^{\mathcal{M}}(\overline{v})$, where $\overline{a} \in M^l$ and α is an ordinal as follows:

$$\phi_{\overline{a},0}^{\mathcal{M}}(\overline{v}) = \bigwedge_{\psi \in X} \psi(\overline{v}),$$

where $X = \{\psi : \mathcal{M} \models \psi(\overline{a})$ and ψ is atomic or the negation of an atomic $\mathcal{L}$-formula$\}$. If α is a limit ordinal, then

$$\phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \bigwedge_{\beta<\alpha} \phi^{\mathcal{M}}_{\overline{a},\beta}(\overline{v}).$$

If $\alpha = \beta + 1$, then

$$\phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \bigwedge_{b\in M} \exists w\ \phi^{\mathcal{M}}_{\overline{a}b,\beta}(\overline{v}, w) \wedge \forall w \bigvee_{b\in M} \phi^{\mathcal{M}}_{\overline{a}b,\beta}(\overline{v}, w).$$

Lemma 2.9 *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures, $\overline{a} \in M^l$, and $\overline{b} \in N^l$. Then, $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ if and only if $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b})$.*

Proof We prove this by induction on α (see Appendix A). Because $(\mathcal{M}, \overline{a}) \sim_0 (\mathcal{N}, \overline{b})$ if and only if they satisfy the same atomic formulas, the lemma holds for $\alpha = 0$.

Suppose that γ is a limit ordinal and the lemma is true for all $\alpha < \gamma$. Then

$$\begin{aligned}
(\mathcal{M}, \overline{a}) \sim_\gamma (\mathcal{N}, \overline{b}) &\Leftrightarrow (\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b}) \text{ for all } \alpha < \gamma \\
&\Leftrightarrow \mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b}) \text{ for all } \alpha < \gamma \\
&\Leftrightarrow \mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\gamma}(\overline{b}).
\end{aligned}$$

Suppose that the lemma is true for α. First, suppose that $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$. Let $c \in M$. Because

$$\mathcal{N} \models \bigwedge_{x\in M} \exists w\ \phi^{\mathcal{M}}_{\overline{a}x,\alpha}(\overline{b}, w),$$

there is $d \in N$ such that $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a}c,\alpha}(\overline{b}, d)$. By induction, $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$. If $d \in N$, then because

$$\mathcal{N} \models \forall w \bigvee_{c\in M} \phi^{\mathcal{M}}_{\overline{a}c,\alpha}(\overline{b}, w)$$

there is $c \in M$ such that $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a}c,\alpha}(\overline{b}, d)$ and $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$. Thus $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{b})$.

Suppose, on the other hand, that $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{b})$. Suppose that $c \in M$, then there is $d \in N$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$ and $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a}c,\alpha}(\overline{b}, d)$. Similarly, if $d \in N$, then there is $c \in M$ such that $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a}c,\alpha}(\overline{b}, d)$. Thus, $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$, as desired.

Lemma 2.10 *For any infinite $\mathcal{L}$-structure $\mathcal{M}$, there is an ordinal $\alpha < |M|^+$ such that if $\overline{a}, \overline{b} \in M^l$ and $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{M}, \overline{b})$, then $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{M}, \overline{b})$ for all β. We call the least such α the* Scott rank *of $\mathcal{M}$.*

Proof Let $\Gamma_\alpha = \{(\overline{a}, \overline{b}) : \overline{a}, \overline{b} \in M^l$ for some $l = 0, 1, \dots$ and $(\mathcal{M}, \overline{a}) \not\sim_\alpha (\mathcal{M}, \overline{b})\}$. Clearly, $\Gamma_\alpha \subseteq \Gamma_\beta$ for $\alpha < \beta$.

Claim 1 If $\Gamma_\alpha = \Gamma_{\alpha+1}$, then $\Gamma_\alpha = \Gamma_\beta$ for all $\beta > \alpha$.

We prove this by induction on β. If β is a limit ordinal and the claim holds for all $\gamma < \beta$, then it also holds for β. Suppose that the claim is true for $\beta > \alpha$ and we want to show that it holds for $\beta+1$. Suppose that $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{M}, \overline{b})$ and $c \in M$. Because $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{M}, \overline{b})$, there is $d \in N$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{M}, \overline{b}, d)$. By our inductive assumption, $(\mathcal{M}, \overline{a}, c) \sim_\beta (\mathcal{M}, \overline{b}, d)$. Similarly, if $d \in M$, then $c \in M$ such that $(\mathcal{M}, \overline{a}, c) \sim_\beta (\mathcal{M}, \overline{b}, d)$. Thus, $(\mathcal{M}, \overline{a}) \sim_{\beta+1} (\mathcal{M}, \overline{b})$ as desired.

Claim 2 There is an ordinal $\alpha < |M|^+$ such that $\Gamma_\alpha = \Gamma_{\alpha+1}$.

Suppose not. Then, for each $\alpha < |\mathcal{M}|^+$, choose $(\overline{a}_\alpha, \overline{b}_\alpha) \in \Gamma_{\alpha+1} \backslash \Gamma_\alpha$. Because $\Gamma_\alpha \subseteq \Gamma_\beta$ for $\alpha < \beta$, the function $\alpha \mapsto (\overline{a}_\alpha, \overline{b}_\alpha)$ is one-to-one. Because there are only $|M|$ finite sequences from M this is impossible.

We conclude this section with Scott's Isomorphism Theorem that every countable $\mathcal{L}$-structure is described up to isomorphism by a single $\mathcal{L}_{\omega_1,\omega}$-sentence.

Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure of cardinality κ, and let α be the Scott rank of $\mathcal{M}$. Let $\Phi^{\mathcal{M}}$ be the sentence

$$\phi^{\mathcal{M}}_{\emptyset,\alpha} \wedge \bigwedge_{l=0}^{\infty} \bigwedge_{\overline{a} \in M^l} \forall \overline{v} (\phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) \rightarrow \phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{v})).$$

Because all of the conjunctions and disjunctions in $\phi^{\mathcal{M}}_{\overline{a},\beta}$ are of size κ, $\phi^{\mathcal{M}}_{\overline{a},\beta} \in \mathcal{L}_{\kappa^+,\omega}$ for all ordinals $\beta < \kappa^+$. Thus $\Phi^{\mathcal{M}}$ is an $\mathcal{L}_{\kappa^+,\omega}$-sentence. We call $\Phi^{\mathcal{M}}$ the *Scott sentence* of $\mathcal{M}$. If $\mathcal{M}$ is countable, then $\Phi^{\mathcal{M}} \in \mathcal{L}_{\omega_1,\omega}$.

Theorem 2.11 (Scott's Isomorphism Theorem) *Let $\mathcal{M}$ and $\mathcal{N}$ be a countable $\mathcal{L}$-structures, and let $\Phi^{\mathcal{M}} \in \mathcal{L}_{\omega_1,\omega}$ be the Scott sentence of $\mathcal{M}$. Then, $\mathcal{N} \cong \mathcal{M}$ if and only if $\mathcal{N} \models \Phi^{\mathcal{M}}$.*

Proof Because α is the Scott rank of $\mathcal{M}$, $\mathcal{M} \models \Phi^{\mathcal{M}}$. An easy induction left to the exercises shows that if $\mathcal{N} \cong \mathcal{M}$, then $\mathcal{M}$ and $\mathcal{N}$ model the same $\mathcal{L}_{\infty,\omega}$-sentences.

On the other hand, suppose that $\mathcal{N}$ models $\Phi^{\mathcal{M}}$. We do a back-and-forth argument to build a sequence of finite partial embeddings $f_0 \subseteq f_1 \subseteq \dots$ from $\mathcal{M}$ to $\mathcal{N}$ such that if $\overline{a}$ is the domain of f_i, then

$$(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, f_i(\overline{a})). \qquad (*)$$

Let $m_0, m_1, \dots$ list M and $n_0, n_1, \dots$ list N.

At stage 0, we let $f_0 = \emptyset$. Because $\mathcal{N} \models \phi^{\mathcal{M}}_{\emptyset,\alpha}$, $\mathcal{M} \sim_\alpha \mathcal{N}$ and $(*)$ holds.

Suppose we are at stage $n+1$. Let $\overline{a}$ be the domain of f_n. Because $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, f(\overline{a}))$, $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha}(f(\overline{a}))$. Because $\mathcal{N} \models \Phi^{\mathcal{M}}$, $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha+1}(f(\overline{a}))$ and $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, f(\overline{a}))$.

If $n+1 = 2i+1$, we want to ensure that m_i is in the domain of f_{n+1}. If m_i is in the domain of f_n, then $f_n = f_{n+1}$. If not, choose $b \in N$ such that $(\mathcal{M}, \overline{a}, m_i) \sim_\alpha (\mathcal{N}, f(\overline{a}), b)$ and extend f_n to f_{n+1} by sending m_i to b.

If $n = 2i + 2$, we want to ensure that n_i is in the image of f_{n+1}. If it is already in the image of f_n, let $f_{n+1} = f_n$. Otherwise, we can find $m \in M$ such that $(\mathcal{M}, \overline{a}, m) \sim_\alpha (\mathcal{N}, f(\overline{a}), n_i)$ and extend f_n to f_{n+1} by simply sending m to n_i.

Corollary 2.12 *Let ϕ be a satisfiable sentence of $\mathcal{L}_{\omega_1,\omega}$. The following are equivalent:*

a) $\phi \models \psi$ or $\phi \models \neg\psi$ for any $\psi \in \mathcal{L}_{\omega_1,\omega}$;

b) $\phi \models \psi$ or $\phi \models \neg\psi$ for any $\psi \in \mathcal{L}_{\infty,\omega}$;

c) ϕ is $\aleph_0$-categorical.

If these equivalent conditions hold we say that ϕ is complete.

Proof b)$\Rightarrow$ a) is clear.

c) $\Rightarrow$ b) by Scott's Theorem.

a) $\Rightarrow$ c) by Downward Löwenheim-Skolem.

Exercise 2.13 Give an example of an $\aleph_1$-categorical $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ that is

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