5 Counterexamples to Vaught's Conjecture

A sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$ is a *counterexample to Vaught's Conjecture*, or simply a *counterexample*, if

$$\aleph_0 < I(\sigma, \aleph_0) < 2^{\aleph_0},$$

i.e., σ has uncountably many countable models but fewer than continuum many models.

Fix σ a counterexample.

A key fact about counterexamples is that there are few types for any countable fragment. Let $\mathcal{F}$ be a fragment. Let

$$S_{\mathcal{F}}(\sigma) = \{\mathrm{tp}^{\mathcal{M}}_{\mathcal{F}}(\overline{a}) : \mathcal{M} \models \sigma, \overline{a} \in M^n \text{ for some } n\}$$

be the set of complete $\mathcal{F}$-types realized in models of σ. We say that a sentence σ is *scattered* if $|S_{\mathcal{F}}(\sigma)| \leq \aleph_0$ for every countable fragment $\mathcal{F}$ with $\sigma \in \mathcal{F}$.

Lemma 5.1 *If σ is a counterexample, then σ is scattered.*

Proof Fix $\mathcal{F}$ a countable fragment. Then

$$S_{\mathcal{F}}(\sigma) = \{p : \exists \mathcal{M} = (\mathbb{N}, \ldots), \mathcal{M} \models \sigma, \exists a \in M^n \ \mathrm{tp}^{\mathcal{M}}(a) = p\}$$

is a $\boldsymbol{\Sigma}^1_1$-set (for details see the proof of Theorem 4.4.12 in [10]) . Thus, if it is uncountable, it has cardinality $2^{\aleph_0}$ (see for example [5] 14.13) But this is only possible if $I(\sigma, \aleph_0) = 2^{\aleph_0}$.

Countable Models

We begin by showing a counterexample has exactly $\aleph_1$ countable models. The proof is exactly the same as the proof for first order theories in S4.4 of [10].

Theorem 5.2 (Morley) *If σ is scattered, then $I(\sigma, \aleph_0) \leq \aleph_1$. Thus if σ is a counterexample, $I(T, \aleph_0) = \aleph_1$.*

Proof The proof uses an analysis of models is similar to Scott's analysis (but faster). We build a sequence of countable fragments $(\mathcal{F}_\alpha : \alpha < \omega_1)$ such that $\sigma \in \mathcal{F}_0$, if $p \in S_{\mathcal{F}_\alpha}(\sigma)$, then $\bigwedge_{\phi \in p} \phi \in \mathcal{F}_{\alpha+1}$ and $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ for α a limit ordinal.

If $\mathcal{M} \models \sigma$, let $\mathrm{tp}^{\mathcal{M}}_\alpha(\overline{a})$ be the $\mathcal{F}_\alpha$-type realized by $\overline{a}$ in $\mathcal{M}$.

For each countable $\mathcal{M} \models \sigma$ there is $\gamma < \omega_1$ such that if $\mathrm{tp}^{\mathcal{M}}_\gamma(\overline{a}) = \mathrm{tp}^{\mathcal{M}}_\gamma(\overline{b})$, then $\mathrm{tp}^{\mathcal{M}}_{\gamma+1}(\overline{a}) = \mathrm{tp}^{\mathcal{M}}_{\gamma+1}(\overline{b})$. We call γ the *height* of $\mathcal{M}$.

Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable models of σ, $\mathcal{M}$ has height γ, and $\mathcal{M} \equiv_{\mathcal{F}_{\alpha+1}} \mathcal{N}$.

Claim $\mathcal{N}$ has height γ.

Suppose $\mathrm{tp}_\gamma^{\mathcal{N}}(\overline{a}) = \mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b})$. Call this type p. Suppose $\psi(\overline{v}) \in \mathcal{F}_{\gamma+1}$ such that $\mathcal{N} \models \psi(\overline{a}) \wedge \neg\psi(\overline{b})$. Then

$$\mathcal{N} \models \exists\overline{v}\exists\overline{w}\big[\big(\bigwedge_{\phi\in p}(\phi(\overline{v}) \wedge \phi(\overline{w}))\big) \wedge \psi(\overline{v}) \wedge \neg\psi(\overline{w})\big].$$

Since this sentence is in $\mathcal{F}_{\gamma+1}$ is also true in $\mathcal{M}$, contradicting that $\mathcal{M}$ has height γ.

Let $P = \{\overline{a} \mapsto \overline{b} : \mathrm{tp}_\gamma^{\mathcal{M}}(\overline{a}) = \mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b})\}$. We claim that P is a back-and-forth system.

Suppose $\mathrm{tp}_\gamma^{\mathcal{M}}(\overline{a}) = \mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b})$ and $c \in M$. Let $p(\overline{v}, w)$ be the $\mathcal{F}_\gamma$-type of $(\overline{a}, c)$. Then

$$\mathcal{M} \models \exists\overline{v}, w \bigwedge_{\phi\in p}\phi(\overline{v}, w).$$

As this is an $\mathcal{F}_{\gamma+1}$-formula, it is true in $\mathcal{N}$. Thus there is $(\overline{b}', d') \in N$ realizing p. Since p extends $\mathrm{tp}_\gamma^{\mathcal{M}}(\overline{a})$, we must have $\mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b}) = \mathrm{tp}_\gamma^{\mathcal{N}}(\overline{b}')$. Since $\mathcal{N}$ has height γ, $\mathrm{tp}_{\gamma+1}^{\mathcal{N}}(\overline{b}) = \mathrm{tp}_{\gamma+1}^{\mathcal{N}}(\overline{b}')$. But

$$\mathcal{N} \models \exists w \bigwedge_{\phi\in p}\phi(\overline{b}', w).$$

As this is an $\mathcal{F}_{\gamma+1}$-formula, there is $d \in \mathcal{N}$ such that $(\overline{b}, d)$ realizes p. Thus $\overline{a}, c \mapsto \overline{b}, d \in P$. The other direction is similar.

Thus if $\mathcal{M} \equiv_{\mathcal{F}_{\gamma+1}} \mathcal{N}$ are countable models of height γ they are isomorphic. If $\mathcal{M}$ is a countable model of σ is determined up to isomorphism by its height γ and $\mathrm{Th}_{\gamma+1}(\mathcal{M}) = \mathrm{tp}_{\gamma+1}^{\mathcal{M}}(\emptyset)$. As there are at most $\aleph_1$ choices for γ and, given γ only countably many choices for its $\mathcal{F}_{\gamma+1}$-theory, $I(\sigma, \aleph_0) \le \aleph_1$.

Our next goal is to show that every counterexample has an uncountable model. The proof uses the idea of "minimal counterexamples" introduced by Harnik and Makkai.

Minimal Counterexamples

Definition 5.3 Suppose σ is a counterexample. We say that σ is a *minimal counterexample* if for every sentence $\psi \in \mathcal{L}_{\omega_1,\omega}$, either $\sigma \wedge \psi$ or $\sigma \wedge \neg\psi$ has at most countably many countable models.

Lemma 5.4 (Harnik-Makkai) *If σ is a counterexample, then there is a minimal counterexample θ with $\theta \models \sigma$.*

Proof

Fix σ a counterexample to Vaught's Conjecture and suppose there is no minimal θ with $\theta \models \sigma$. The basic idea is that we will build a tree of counterexamples $(\sigma_\tau : \tau \in 2^{<\omega})$ such that:

i) $\sigma_\emptyset = \sigma$;

ii) $\sigma_\eta \models \sigma_\tau$ for $\tau \subseteq \eta$;

iii) $\sigma_{\tau,0} \wedge \sigma_{\tau,1}$ is unsatisfiable.

This is easy to do. At any stage τ, σ_τ is a non-minimal counterexample. Thus there is a ψ such that $\sigma_{\tau,0} = \sigma_\tau \wedge \psi$ and $\sigma_{\tau,1} = \sigma_\tau \wedge \neg\psi$ are both counterexamples.

We would like to get a contradiction by considering

$$T_f = \{\sigma_{f|n} : n \in \omega\}$$

for $f \in 2^\omega$. If each T_f is satisfiable, we could easily conclude that σ has $2^{\aleph_0}$ models, a contradiction. The problem is that, in the absence of compactness, we have no guarantee that T_f is satisfiable. We will need to exercise more care.

Add C a countable set of new constant symbols. Let $\Sigma = \{s : s$ is a finite set of $\mathcal{L}_{\omega_1,\omega}$-sentences using only finitely many constants from C, such that $s \cup \{\sigma\}$ has uncountably many countable models$\}$.

Claim Σ is a consistency property.

Let's check (C4). Suppose $\displaystyle\bigvee_{\phi \in X} \phi \in s \in \Sigma$. Since there are uncountably many countable models of s. By the Pigeonhole-Principle, there are uncountably many countable models of $s \cup \{\psi\}$ for some $\psi \in X$. Then $s \cup \{\psi\} \in \Sigma$.

Suppose s is a finite set of $\mathcal{L}_{\omega_1,\omega}$-sentences with only finitely many constants from C. Let $\theta(\overline{c})$ be the conjunction of all sentences in s. In any countable $\mathcal{L}$-structure there are only countably many ways to interpret the constants $\overline{c}$. Thus $s \in \Sigma$ if and only if $\sigma \wedge \exists \overline{v} \theta(\overline{v})$ is a counterexample.

We will build a sequence of countable fragments $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$ and $\mathcal{F} = \bigcup \mathcal{F}_n$. Let $\phi_0, \phi_1, \ldots,$ list all $\mathcal{F}$-sentences and let $t_0, t_1, \ldots$ list all basic $\mathcal{F}$-terms, both lists should have list each item infinitely often.

We will also build a tree $(s_\sigma : \sigma \in 2^{<\omega})$ such that:

i) $s_\emptyset = \{\sigma\}$ and each $s_\sigma \in \Sigma$;

ii) $s_\eta \subseteq s_\tau$ for $\eta \subset \tau$;

iii) for each τ, there is an $\mathcal{L}_{\omega_1,\omega}$ sentence ψ with no constants from C such that $\psi \in s_{\tau,0}$ and $\neg\psi \in s_{\tau,1}$;

iv) for $i = 0, 1$, if $|\tau| = n$ and $s_\tau \cup \{\phi_n\} \in \Sigma$, then $\phi_n \in \sigma_{\tau,i}$, moreover if, in addition, $\phi_n = \bigvee_{\psi \in X} \psi$, then $\psi \in \sigma_{\tau,i}$ for some $\sigma \in X$, and if $\phi_n = \exists v\ \psi(v)$, then $\psi(c) \in \sigma_{\tau,i}$ for some $c \in C$;

v) for $i = 0, 1$, if $|\tau| = n$, there is $c \in C$ such that $t_n = c \in s_{\tau,i}$.

As in the proof of the Model Existence Theorem, conditions iv) and v) will insure that $T_f = \bigcup s_{f|n}$ is satisfiable for all $f \in 2^\omega$. As above, condition iii) will insure that if $\mathcal{M} \models T_f$ and $\mathcal{N} \models T_g$, then their reducts to the original language are non-isomorphic.

Let $\mathcal{F}_0$ be a countable fragment containing σ. In general, $\mathcal{F}_{n+1}$ will be a countable σ-fragment, containing $\mathcal{F}_n \cup \displaystyle\bigcup_{|\tau| \leq n} s_\tau$. Although our fragment increases in the construction, it is easy to build listings $\phi_0, \phi_1, \ldots$ and $t_0, t_1, \ldots$. The only additional condition we need is that $\phi_n \in \mathcal{F}_n$ and t_n is an $\mathcal{F}_n$-term.

Let $s_\emptyset = \{\sigma\}$.

Suppose we are given s_τ where $|\tau| = n$. As in the proof of the Model Existence Theorem, it is easy to find $s'_\tau \in \Sigma$ such that $s_\tau \subseteq s'_\tau$:

i) if $s_\tau \cup \{\phi_n\} \in \Sigma$, then $\phi_n \in s'_\tau$, moreover, in that case if

a) if $\phi_n = \bigvee_{x \in X} \psi$, then $\psi \in s'_\tau$ for some $\psi \in X$, and

b) if $\phi_n = \exists v\ \psi(v)$, then $\psi(c) \in s'_\tau$ for some $c \in C$;

ii) $t_n = c \in s'_\tau$ for some $c \in C$.

Let $\theta(\overline{c})$ be the conjunction of all formulas in s'_τ. As remarked above, $\sigma \wedge \exists \overline{v}\ \theta(\overline{v})$ is a counterexample. Since it is not minimal, there is and $\mathcal{L}_{\omega_1,\omega}$-sentence ψ (with no constants from C) such that $s_{\tau,0} = s'_\tau \cup \{\psi\}$ and $s_{\tau_1,/}$

This completes the construction and the proof.

We will need a refinement of this result. Let σ be a minimal counterexample. Suppose $\phi(\overline{v})$ is an $\mathcal{L}_{\omega_1,\omega}$-formula, we say that ϕ is *σ-large* if there are uncountably many countable models of $\sigma \wedge \exists \overline{v}\ \phi(\overline{v})$. We say that a σ-large formula $\phi(\overline{v})$ is *minimal σ-large* if for all $\mathcal{L}_{\omega_1,\omega}$-formulas $\psi(\overline{v})$ exactly one of $\phi \wedge \psi$ and $\phi \wedge \neg\psi$ is σ-large.

Corollary 5.5 *If $\phi(\overline{v})$ is σ-large, then there is a minimal σ-large formula $\theta(\overline{v})$ with $\theta(\overline{v}) \models \phi(\overline{v})$.*

Proof If $\phi(\overline{v})$ has free variables $v_1, \ldots, v_n$, add constants $d_1, \ldots, d_n$ to $\mathcal{L}$ and apply the theorem to $\sigma \wedge \phi(\overline{d})$. There is $\psi(\overline{d})$ a minimal counterexample with $\psi(\overline{d}) \models \sigma \wedge \phi(\overline{d})$. Then $\psi(\overline{v})$ is a minimal σ-large formula with $\psi(\overline{v}) \models \phi(\overline{v})$.

Digression: prime models

We quickly review some material on atomic and prime models from [6]. Fix $\mathcal{F}$ a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and T be an $\mathcal{F}$-complete theory.

Definition 5.6 We say $\phi(\overline{v})$ is complete if it is satisfiable and for all formulas $\psi(\overline{v})$ in $\mathcal{F}$ either

$$T \models \phi(\overline{v}) \rightarrow \psi(\overline{v}) \text{ or } T \models \phi(\overline{v}) \rightarrow \neg\psi(\overline{v}).$$

We say that T is *$\mathcal{F}$-atomic* if every satisfiable $\mathcal{F}$ formula is a T-consequence of a complete formula.

We say that $\mathcal{M} \models T$ is $\mathcal{F}$-atomic if every $\overline{a} \in M^n$ satisfies a complete formula.

We say that $\mathcal{M} \models T$ is $\mathcal{F}$-prime if there is an $\mathcal{F}$-elementary embeding of $\mathcal{F}$ into any $\mathcal{N} \models T$.

Theorem 5.7 *If T has only countably many $\mathcal{F}$-types, then there is $\mathcal{M} \models T$ that is $\mathcal{F}$-atomic and $\mathcal{F}$-prime.*

Proof Use the Omitting Types Theorem to build $\mathcal{M} \models T$ omitting all types not containing a complete formula. Clearly $\mathcal{M}$ is $\mathcal{F}$-atomic. The usual proof that countable atomic models of first order theories are prime, adapts immediately to prove that $\mathcal{M}$ is $\mathcal{F}$-prime.

Uncountable models of counterexamples

We now fix σ a minimal counterexample.

Our proof will need fragments with extra closure properties. We say that $\mathcal{F}$ is *rich* if and for all $\phi(\overline{v})$ is σ-large, there is a σ-minimal $\psi(\overline{v}) \in \mathcal{F}$ with $\sigma \models \psi(\overline{v}) \rightarrow \phi(\overline{v})$. By Corollary 5.5, if $\mathcal{F}$ is a countable fragment we can find $\mathcal{F}' \supseteq \mathcal{F}$ that is countable and rich. If $\mathcal{F}$ is a countable rich fragment of $\mathcal{L}_{\omega_1,\omega}$, let

$T_{\mathcal{F}} = \{\psi \in \mathcal{F} : \psi$ a sentence such that $\sigma \wedge \psi$ has uncountably many countable models$\}$. First note that $T_{\mathcal{F}}$ is a complete $\mathcal{F}$-theory. Moreover, is $\phi \in \mathcal{F} \setminus T_{\mathcal{F}}$, then $\sigma \wedge \phi$ has only countably many countable models. Thus $T_{\mathcal{F}}$ is satisfiable, indeed it has uncountably many countable models.

If $\phi(\overline{v})$ is a σ-minimal formula, then for any formula $\psi(\overline{v})$,

$$T_{\mathcal{F}} \models \phi(\overline{v}) \rightarrow \psi(\overline{v}) \text{ or } T_{\mathcal{F}} \models \phi(\overline{v}) \rightarrow \neg\psi(\overline{v}).$$

Thus every σ-minimal formula is $\mathcal{F}$-complete, and every $\mathcal{F}$-formula is applied by an σ-minimal formula. Thus $T_{\mathcal{F}}$ is $\mathcal{F}$-atomic.

We now describe our basic construction. For each $\alpha < \omega_1$ we build

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_\alpha \subseteq \ldots.$$

Let $\mathcal{F}_0$ be a countable rich fragment containing σ.

Let $\mathcal{M}_0$ be the prime model of $T_{\mathcal{F}_0}$.

Given $\mathcal{M}_\alpha$ and $\mathcal{F}_\alpha$, let $\mathcal{F}_{\alpha+1}$ be a countable rich fragment containing $\mathcal{F}_\alpha$ and the Scott sentence of $\mathcal{M}_\alpha$. Then let $\mathcal{M}_{\alpha+1}$ be the prime model of $T_{\mathcal{F}_{\alpha+1}}$. Note that $\mathcal{M}_\alpha \prec_{\mathcal{F}_\alpha} \mathcal{M}_{\alpha+1}$.

For β a limit ordinal, let $\mathcal{F}_\beta = \bigcup_{\alpha<\beta} \mathcal{F}_\beta$ and let $\mathcal{M}_\beta = \bigcup_{\alpha<\beta} \mathcal{M}_\alpha$. For all $\overline{a} \in M_\beta$ there is a σ-large $\phi(\overline{v})$ such that $\phi(\overline{a})$. But then $\phi(\overline{v})$ is $\mathcal{F}_\beta$-complete. Thus $\mathcal{M}_\beta$ is the prime model of $T_{\mathcal{F}_\beta}$.

Since the Scott sentence of $\mathcal{M}_\alpha$ is in $\mathcal{F}_\beta$ for $\beta > \alpha$ and this sentence is not σ-large, $\mathcal{M}_\alpha \not\cong \mathcal{M}_\beta$ for $\alpha \neq \beta$.

Let $\mathcal{N} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. Then $\mathcal{N} \models \sigma$. Suppose ϕ is an $\mathcal{L}$-sentence such that $\mathcal{N} \models \phi$. Then $\{\alpha < \omega_1 : \mathcal{M}_\alpha \models \phi\}$ is closed unbounded. *(prove this???)* Since the $\mathcal{M}_\alpha$ are not isomorphic, ϕ is σ-large. In particular, ϕ is not the Scott sentence of any model of σ. We have proved the following theorem

Theorem 5.8 *If σ is a counterexample to Vaught's Conjecture, then σ has a model of cardinality $\aleph_1$ that is not $\mathcal{L}_{\omega_1,\omega}$-equivalent to a countable model.*

Further Results

The model of size $\aleph_1$ that we constructed above is not *li*-equivalent to a countable model.

Theorem 5.9 (Harnik-Makkai) *If σ is a counterexample, then there are models of size $\aleph_1$ that are $\mathcal{L}_{\infty,\omega}$-equivalent to countable models.*

The original proof used admissible model theory, Baldwin gave a proof avoiding admissibility. The arguments given in Baldwin's lecutres show that indeed there are $\aleph_1$ models of size $\aleph_1$ that are $\mathcal{L}_{\infty,\omega}$-equivalent to countable models but pairwise $\mathcal{L}_{\infty,\omega}$-inequivalent.

Theorem 5.10 (Harrington) *For all $\alpha < \omega_2$ there is a model of size $\aleph_1$ with Scott rank at least α.*

I have never seen Harrington's proof.

Baldwin noted that using two deep theorems of Shelah's every first order counterexample T has $2^{\aleph_1}$ models of size $\aleph_1$. A first order counterexample T must be non-ω-stable and every non-ω-stable theory has $2^{\aleph_1}$-models of cardinality $\aleph_1$.

Questions 1) Does Harrington's results follow from Sacks' recent paper?
2) Give a proof of Harrington's result without admissibility.
3) Can we prove that $I(\sigma, \aleph_1) = 2^{\aleph_1}$ for all counterexamples σ?
4) Can we find $2^{\aleph_1}$ models that are $\mathcal{L}_{\infty,\omega}$-equivalent to a countable models?

Theorem 5.11 (Hjorth) *There is a counterexample σ with no models of size $\aleph_2$.*

Hjorth's surprising proof used descriptive set theory of the dynamics of Polish group actions.

Question 5 Give a model theoretic proof of Hjorth's theorem.