Manin Kernels

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In his proof of the Mordell conjecture for function fields, Manin [M] showed that if A is an abelian variety defined over a differential field, then there is a homomorphism μ of differential algebraic groups from A into a vector group. The kernel $A^\#$ of μ plays an important role in both Buium's and Hrushovski's proofs ([B] and [H]) of the Mordell-Lang conjecture for characteristic zero function fields. My goal in this note is to give an exposition of one construction of the Manin kernel using the notion of prolongations of algebraic varieties. The treatment of prolongations that I give here takes a rather naive point of view compared to the scheme theoretic treatment in [B]. For smooth varieties the two treatments agree, while for general varieties our prolongations correspond to the reduced points of those of [B]. Section §3 discusses Kolchin's logarithmic-derivative map for algebraic groups defined over the constant field. Though in the spirit of the main theme, this material in not necessary for later sections.

Manin kernels are of great model theoretic interest. If A is a simple abelian variety that does not descend to the constants, then Hrushovski and Sokolovic [HS] showed that $A^\#$ is a locally modular strongly minimal set. In §5 I will discuss this proof.

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Let K be a differentially closed field of characteristic zero with derivation $\delta : K \to K$. Let $k = \{x \in K : \delta(x) = 0\}$ be the field of constants of δ. For convenience, at one point I will assume that K is $\aleph_0$-saturated, but this assumption can be easily avoided.

For $p \in K[X_1, \ldots, X_n]$, let p^δ be the polynomial obtained by differentiating the coefficients of p. If $a = (a_1, \ldots, a_n) \in K^n$, then

$$\delta(p(a)) = \sum_{i=1}^{n} \frac{\partial p}{\partial X_i}(a)\delta(a_i) + p^\delta(a).$$

For $a \in K^n$, we let $\delta(a)$ denote $(\delta(a_1), \ldots, \delta(a_n))$.

Let $K\{X_1, \ldots, X_n\}$ denote the ring of differential polynomials over K. We endow K^n with the *Kolchin topology* or *δ-topology* by taking zero sets of finite systems of differential polynomial equations as basic closed sets. We will use two basic fact about the Kolchin topology. See [K] or [Mr1] for details.

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Differential Basis Theorem There are no infinite descending chains of Kolchin closed sets in K^n.

Kolchin's Irreducibility Theorem If $V \subseteq K^n$ is an irreducible algebraic variety, then V is still irreducible in the Kolchin topology.

§1 Prolongations

Let $V \subseteq K^n$ be an irreducible algebraic variety and let $I(V)$ be the prime ideal of polynomials over K that vanish on V. Let $p_1, \ldots, p_m$ generate $I(V)$.

(1.1) The *prolongation* of V is the variety

$$\begin{aligned}\tau(V) &= \{(a,u) \in K^{2n} : a \in V, \sum_{i=1}^{n} \frac{\partial p}{\partial X_i}(a)u_i + p^{\delta}(a) = 0 \text{ for } p \in I(V)\} \\ &= \{(a,u) \in K^{2n} : a \in V, \sum_{i=1}^{n} \frac{\partial p_j}{\partial X_i}(a)u_i + p_j^{\delta}(a) = 0 \text{ for } j = 1, \ldots, m\}.\end{aligned}$$

Let $\pi : \tau(V) \to V$ be the projection onto the first n-coordinates.

For $a \in V$, let $\nabla_V(a) = (a, \delta(a))$. If no confusion arises we will write ∇ for ∇_V. Since $\delta(p(a)) = 0$ for $a \in V$ and $p \in I(V)$, $\nabla : V \to \tau(V)$ is a section of π (i.e. $\pi \circ \nabla(a) = a$ for $a \in V$).

The prolongation is a twisted version of the *tangent bundle*

$$T(V) = \{(a,u) \in K^{2n} : a \in V, \sum_{i=1}^{n} \frac{\partial p}{\partial X_i}(a)u_i = 0 \text{ for } p \in I(V)\}.$$

Let $s : T(V) \to V$ be the natural projection. The natural section of s is the zero map $z : V \to T(V)$ where $z(a) = (a, 0)$.

Note that if V is defined over k, then $T(V) = \tau(V)$.

(1.2) Suppose X, Y and V are algebraic varieties, $f : X \to V$ and $g : Y \to V$ are regular maps and $f^{-1}(a)$ is a group for $a \in V$. We say that Y is a *torsor* under X, if there is a regular $p : X \times_V Y \to Y$, such that p induces a regular action of $f^{-1}(a)$ on $g^{-1}(a)$ for all $a \in V$.

For $a \in V$, the *tangent space* of V at a is $T_a(V) = \{u \in K^n : (a,u) \in T(V)\}$. Then $T_a(V)$ acts regularly on $\tau_a(V) = \{u : (a,u) \in \tau(V)\}$ by $(u,v) \mapsto u + v$, making $\tau(V)$ a torsor under $T(V)$.

(1.3) In general $\tau(V)$ and $T(V)$ will not be isomorphic as algebraic varieties, but the map $(a,u) \mapsto (a, u + \delta(a))$ is a differential algebraic isomorphism between $T(V)$ and $\tau(V)$.

(1.4) If V is smooth, then $T(V)$ and $\tau(V)$ are smooth.

The Jacobian at $(a,u) \in T(V)$ is

$$\begin{pmatrix} \frac{\partial p_1}{\partial X_1}(a) & \dots & \frac{\partial p_1}{\partial X_n}(a) & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ \frac{\partial p_m}{\partial X_j}(a) & \dots & \frac{\partial p_m}{\partial X_n}(a) & 0 & \dots & 0 \\ \sum_{j=1}^n \frac{\partial^2 p_1}{\partial X_j X_1}(a)u_j & \dots & \sum_{j=1}^n \frac{\partial^2 p_1}{\partial X_j X_n}(a)u_j & \frac{\partial p_1}{\partial X_1}(a) & \dots & \frac{\partial p_1}{\partial X_n}(a) \\ \vdots & & & & & \vdots \\ \sum_{j=1}^n \frac{\partial^2 p_m}{\partial X_j X_1}(a)u_j & \dots & \sum_{j=1}^n \frac{\partial^2 p_m}{\partial X_j X_n}(a)u_j & \frac{\partial p_m}{\partial X_1}(a) & \dots & \frac{\partial p_m}{\partial X_n}(a) \end{pmatrix}.$$

If V is smooth of dimension d, then $T_a(V)$ has dimension d and $T(V)$ has dimension $2d$. Thus the Jacobian matrix above has rank at most $2n-2d$. But

$$\begin{pmatrix} \frac{\partial p_1}{\partial X_1}(a) & \dots & \frac{\partial p_1}{\partial X_n}(a) \\ \vdots & & \vdots \\ \frac{\partial p_m}{\partial X_1}(a) & \dots & \frac{\partial p_m}{\partial X_n}(a) \end{pmatrix}$$

has rank $n-d$. Thus the Jacobian matrix above has rank at least $2n-2d$. Hence $T(V)$ is smooth of dimension $2d$.

A similar argument works for $\tau(V)$.

(1.5) If V is smooth, then $T(V)$ and $\tau(V)$ are irreducible.

Suppose $T(V) = X \cup Y$ where X and Y are Zariski closed proper subsets of $T(V)$. Since $T(V)$ is smooth $X \cap Y = \emptyset$. Since each $T_a(V)$ is irreducible each fiber of s is either contained in X or contained in Y. Since X and Y are proper subsets of $T(V)$, there are $a, b \in V$ such that $s^{-1}(a) \cap X = \emptyset$ and $s^{-1}(b) \cap Y = \emptyset$. But then $V = z^{-1}(X) \cup z^{-1}(Y)$ and $z^{-1}(X)$ and $z^{-1}(Y)$ are proper Zariski closed subsets of V, contradicting the irreducibility of V.

The same argument shows that if $\tau(V) = X \cup Y$, then $V = \nabla^{-1}(X) \cup \nabla^{-1}(Y)$. In this case V is reducible as a differential algebraic variety. This contradicts Kolchin's Irreducibility Theorem.

(1.6) If V is smooth, then $\tau(V)$ is the Zariski closure of $\nabla(V)$ in K^{2n}.

Suppose V has dimension d. Without loss of generality we may assume that the projection map

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_d)$$

from V to K^d is generically surjective. Choose $x = (x_1, \dots, x_n) \in V$ such that $x_1, \dots, x_d$ are differentially independent over F the field of definition of V (this is possible by $\aleph_0$-saturation). Then $x_1, \dots, x_d, \delta(x_1), \dots, \delta(x_d)$ are algebraically independent over F. Since $(x, \delta(x)) \in \nabla(V) \subseteq \tau(V)$, the Zariski closure of $\nabla(V)$ has dimension at least $2d = \dim \tau(V)$. Since $\tau(V)$ is irreducible, it is the Zariski closure of $\nabla(V)$.

Suppose $W \subseteq K^m$ is an algebraic variety and $\phi : V \to W$ is a regular map. Let $\phi(x) = (\phi_1(x), \dots, \phi_m(x))$ where $\phi_i : K^n \to K^m$. For $a \in V$, there is a linear map

$d\phi_a : T_a(V) \to T_{\phi(a)}(W)$

$$d\phi_a u = \Big(\sum_{i=1}^{n} \frac{\partial \phi_1(a)}{\partial X_i} u_i, \ldots, \sum_{i=1}^{n} \frac{\partial \phi_m(a)}{\partial X_i} u_i \Big).$$

Let $\phi^\delta(x) = (\phi_1^\delta(x), \ldots, \phi_m^\delta(x))$.

Let $T(\phi) : T(V) \to T(W)$ by

$$T(\phi)(a, u) = (\phi(a), d\phi_a(u))$$

and $\tau(\phi) : \tau(V) \to \tau(W)$ by

$$\tau(\phi)(a, u) = (\phi(a), d\phi_a(u) + \phi^\delta(a)).$$

(1.7) Suppose $\phi : U \to V$ and $\psi : V \to W$ are regular maps, then $T(\psi \circ \phi) = T(\psi) \circ T(\phi)$ and $\tau(\psi \circ \phi) = \tau(\psi) \circ \tau(\phi)$. Thus T and τ are functors from the category of irreducible affine varieties with regular maps to itself.

Since $d(\psi \circ \phi)_a = d\psi_{\phi(a)} \circ d\phi_a$, $T(\psi \circ \phi) = T(\psi) \circ T(\phi)$.

Since

$$\tau(\psi)(\phi(a), d\phi_a u + \phi^\delta(a)) = (\psi \circ \phi(a), d\psi_{\phi(a)}(d\phi_a u) + d\psi_{\phi(a)}\phi^\delta(a) + \psi^\delta(\phi(a)))$$

it suffices to show:

claim $(\psi \circ \phi)^\delta(a) = d\psi_{\phi(a)}\phi^\delta(a) + \psi^\delta(\phi(a))$.

On one hand

$$\delta(\psi(\phi(a))) = d(\psi \circ \phi)_a \delta(a) + (\psi \circ \phi)^\delta(a).$$

While on the other hand

$$\begin{aligned} \delta(\psi(\phi(a)) &= d\psi_{\phi(a)}\delta(\phi(a)) + \psi^\delta(\phi(a)) \\ &= d\psi_{\phi(a)}(d\phi_a \delta(a) + \phi^\delta(a)) + \psi^\delta(\phi(a)) \\ &= d(\psi \circ \phi)_a \delta(a) + d\psi_{\phi(a)}\phi^\delta(a) + \psi^\delta(\phi(a)) \end{aligned} \quad .$$

The claim follows.

(1.8) Suppose $\phi : V \to W$ is a regular map. Then $\tau(\phi) \circ \nabla_V = \nabla_W \circ \phi$.

Note that

$$\tau(\phi)(a, \delta(a)) = (\phi(a), d\phi_a \delta(a) + \phi^\delta(a))$$

while

$$\nabla_W(\phi(a)) = (\phi(a), \delta(\phi(a))) = (\phi(a), d\phi_a(\delta(a)) + \phi^\delta(a)).$$

(1.9) **Patching**: By a *prevariety* we mean a topological space W with a finite open cover $W = W_1 \cup \ldots \cup W_m$ and homeomorphisms $f_i : V_i \to W_i$ where V_i is an affine algebraic variety such that:

i) if $U_{i,j} = f_i^{-1}(W_i \cap W_j)$, then $U_{i,j}$ is a Zariski open subset of V_i, and
ii) if $f_{i,j} = f_j \circ f_i^{-1} : U_{i,j} \to U_{j,i}$, then $f_{i,j}$ is regular.

A *variety* is prevariety W where $\Delta_W = \{(x,y) \in W^2 : x = y\}$ is closed

Suppose W is a variety and W_i, V_i and f_i are as above. We define the prolongation $\pi : \tau(W) \to W$ as follows. Let $g_{i,j} : \tau(U_{i,j}) \to \tau(U_{j,i})$ be $\tau(f_{i,j})$. Let X be the disjoint union of the $\tau(V_i)$. On X we have an equivalence relation E where xEy if and only if $g_{i,j}(x) = y$ for some i and j. Let $\tau(W)$ be X/E with the induced topology. If $x \in \tau(V_i)$ let $\pi(x) = f_i(\pi(x))$. Clearly $\tau(W)$ is a prevariety. We need to show that $\Delta_{\tau(W)}$ is closed in $\tau(W) \times \tau(W)$.

Suppose $x, y \in \tau(W)$ and $x \neq y$. If $\pi(x) \neq \pi(y)$ then there is an open U in W^2 such that $(x,y) \in U$ and $U \cap \Delta_W = \emptyset$. The $\pi^{-1}(U) \cap (x,y) \in U$ and $\pi^{-1}(U) \cap \Delta_{\tau(W)} = \emptyset$ so (x,y) is not in the closure of $\Delta_{\tau(W)}$.

If $\pi(x) = \pi(y)$ we can find V_i and $a, b \in V_i$ such that $x = a/E$ and $y = b/E$. There is $U \subset \tau(V_i)^2$ such that $(a,b) \in U$ and $U \cap \Delta_{\tau(V_i)} = \emptyset$. But then U/E shows that (x,y) is not in the closure of $\Delta_{\tau(W)}$.

A similar construction allows us to define the tangent bundle of W.

By (1.8)

$$\nabla \circ f_{i,j} = \tau(f_{i,j}) \circ \nabla.$$

This allows us to define $\nabla : W \to \tau(W)$ by $\nabla(x) = \nabla(f_i^{-1}(x))/E$.

(1.10) If V and W are varieties, then it is easy to see that $T(V \times W)$ is naturally isomorphic to $T(V) \times T(W)$. The same is true for prolongations.

§2 Prolongations of algebraic groups

Suppose G is an algebraic group with multiplication m. We will write ab for $m(a,b)$.

(2.1) $T(G)$ and $\tau(G)$ are algebraic groups with multiplication $T(m)$ and $\tau(m)$ respectively.

This follows from (1.7) and (1.10). For example, for associativity it is enough to apply T or τ to the commutative diagram

$$\begin{array}{ccc} (a,b,c) & \mapsto & (ab,c) \\ \downarrow & & \downarrow \\ (a,bc) & \mapsto & a(bc) = (ab)c. \end{array}$$

Similar arguments work for inverse.

There is a useful explicit formula for the group law on the tangent bundle. We view G as a multiplicative group. For $g \in G$, let $\lambda^g, \rho^g, i^g : G \to G$ be given by:

$$\lambda^g(x) = gx, \rho^g(x) = xg, \text{ and } i^g(x) = g^{-1}xg.$$

The group law on $T(G)$ is given by

$$(g,u)(h,v) = (gh, d\lambda^g_h v + d\rho^h_g u).$$

Let e be the identity element of G. There is a natural right action of G on $T_e(G)$ given by $a * g = di^g_e a$. Note that $a * gh = (a * g) * h$.

(2.2) With respect to this action there is a crossed group homomorphism $\eta : T(G) \to T_e(G)$.
Since $\lambda_{g^{-1}}(g) = e$, $d\lambda_g^{g^{-1}} : T_g(G) \to T_e(G)$. Define $\eta(g, u) = d\lambda_g^{g^{-1}} u$. Then

$$\begin{aligned}\eta((g, u)(h, v)) &= d\lambda_{gh}^{(gh)^{-1}}(d\lambda_h^h v + d\rho_g^h u)\\ &= d\lambda_h^{h^{-1}} v + (di^h)_e(d\lambda_g^{g^{-1}} u)\\ &= \eta(h, v) + \eta(g, u) * h.\end{aligned}$$

(2.3) The tangent bundle $T(G)$ is isomorphic to $G \ltimes T_e(G)$, the semidirect product of G and $T_e(G)$ with respect to this action.
The isomorphism is given by $(g, u) \mapsto (g, \eta(g, u))$. Note that

$$\begin{aligned}(g, u)(h, v) &\mapsto (gh, \eta(gh, d\lambda_h^g v + d\rho_g^h u))\\ &= (gh, d\lambda_{gh}^{(gh)^{-1}}(d\lambda_h^g v + d\rho_g^h u))\\ &= (gh, d\lambda_h^{h^{-1}} v + di_e^h(d\lambda_e^g u))\\ &= (g, \eta(g, u))(h, \eta(h, v)) \text{ under the multiplication of } G \ltimes T_e(G).\end{aligned}$$

(2.4) If G is a commutative algebraic group, then $\eta : T(G) \to T_e(G)$ is a group homomorphism and $T(G) \cong G \oplus T_e(G)$.

(2.5) The differential section $\nabla : G \to \tau(G)$ is a group homomorphism.
If m is the multiplication of G, then, by 1.8,

$$\tau(m) \circ \nabla_{G \times G} = \nabla_G \circ m.$$

§3 The logarithmic-derivative

For this section G is a commutative algebraic group of dimension d defined over the field of constants k. Since G is defined over k, $T(G) = \tau(G)$. We will exploit the fact that, in this case, ∇ is a nontrivial section of the tangent bundle.

Define the *logarithmic-derivative* $l : G \to T_e(G)$ by $l = \eta \circ \nabla$. The logarithmic derivative is a group homomorphism from G to $T_e(G)$.

(3.1) **Example**: Let G be the additive group $(K, +)$. In this case $d\lambda_g^{g^{-1}}(u) = u$ and $l(g) = \delta(g)$.

(3.2) **Example**: Let G be the multiplicative group $(K^*, \cdot)$. In this case $d\lambda_g^{g^{-1}} u = u/g$ and $l(g) = \delta(g)/g$.

(3.3) The logarithmic-derivative is surjective.

Suppose $\alpha \in T_e(G)$. Let $W = \{(g,u) \in T(G) : d\lambda_g^{g^{-1}} u = \alpha\}$. Since $d\lambda_g^{g^{-1}}$ is an isomorphism between $T_g(G)$ and $T_e(G)$, W projects onto G. By the following theorem of Pierce and Pillay [PP] there is $g \in G$ such that $(g, \delta(g)) \in W$.

(3.4) **Geometric Axioms for DCF**: Let K be a differential field. The following are equivalent:

i) K is differentially closed.

ii) If V is an algebraic variety defined over K, $W \subseteq \tau(V)$ is constructible and W projects generically onto V, then there is $a \in V$ such that $\nabla(a) \in W$.

(3.5) Let $s_1, \ldots, s_d : T_e(G) \to K$ be linearly independent and let $\chi_i : G \to K$ be $\chi_i = s_i \circ l$. Since l is surjective, $\chi_1, \ldots, \chi_d$ are linearly independent differential algebraic homomorphisms from G into the additive group. We call such homomorphisms *differential additive characters.*

For the rest of this section we will look at the relationship between the logarithmic-derivation and invariant differential forms.

If V and W are algebraic varieties, $f : V \to W$ is a regular map and ω is a regular differential form on W, we let $f_*\omega$ be the regular differential form on V given by

$$f_*\omega_p(u) = \omega_{f(p)}(df_p u)$$

for $u \in T_p(V)$. We say that a regular differential form ω is an *invariant differential form* if $\lambda_*^g \omega = \omega$ for all $g \in G$. In other words

$$\omega_{gh}(u) = \omega_h(d\lambda_{gh}^{g^{-1}} u) = \omega_g(d\lambda_{gh}^{h^{-1}} u).$$

If G is a d-dimensional algebraic group, then the invariant differential forms are an d-dimensional K-vector space.

(3.6) If ω is an invariant differential form on G, define $\chi : G \to K$ by $\chi_\omega(g) = \omega_g \nabla(g)$. Then χ_ω is a differential algebraic additive character.

By (2.5)

$$\nabla(gh) = d\lambda_h^g \nabla(h) + d\lambda_g^h \nabla(g).$$

Thus

$$\begin{aligned}
\chi_\omega(gh) &= \omega_{gh} \nabla(gh) \\
&= \omega_{gh}(d\lambda_g^h \nabla(g) + d\lambda_h^g \nabla(h)) \\
&= \omega_{gh}(d\lambda_g^h \nabla(g)) + \omega_{gh}(d\lambda_h^g \nabla(h)) \\
&= \omega_g \nabla(g) + \omega_h \nabla(h) \text{ by the invariance of } \omega \\
&= \chi_\omega(g) + \chi_\omega(h).
\end{aligned}$$

(3.7) **Example**: Suppose $\delta(a) = 0$ and E is the elliptic curve with affine part $Y^2 = X(X-1)(X-a)$. In this case $\omega = \dfrac{dx}{y}$ is an invariant differential. This gives rise to the homomorphism

$$\chi(x,y) = \frac{\delta(x)}{y}$$

from E to K.

(3.8) Let s_i and χ_i be as in (3.5). Let ω^i be the regular differential form defined by

$$\omega^i_g(u) = s_i(d\lambda_g^{g^{-1}} u).$$

Then ω is an invariant form, $\chi_i = \chi_{\omega^i}$ and $\chi_1, \ldots, \chi_d$ spans the space of differential additive characters induced by invariant differentials.

$$\begin{aligned}
\omega^i_{gh}(u) &= s_i(d\lambda_{gh}^{(gh)^{-1}} u) \\
&= s_i(d\lambda_g^{g^{-1}}(d\lambda_{gh}^{h^{-1}} u)) \\
&= \omega^i_g(d\lambda_{gh}^{h^{-1}} u).
\end{aligned}$$

So ω^i is invariant.

$$\chi_i(g) = s_i(l(g)) = s_i(d\lambda_g^{g^{-1}} \nabla(g)) = \omega^i_g(\nabla)(g) = \chi_{\omega^i}(g).$$

Since $\chi_1, \ldots, \chi_d$ are linearly independent they span the space of differential additive characters induced by invariant differentials.

§4 Manin kernels of abelian varieties

For algebraic groups not defined over the constants we do not, in general, have the luxury of an algebraic sections of the prolongation. For abelian varieties A we will get around this by finding an extension $p : \widehat{A} \to A$ where there is an algebraic section of $\pi : \tau(\widehat{A}) \to \widehat{A}$.

Let G and H be commutative algebraic groups. We say that H is an *extension of* G *by a vector group* if there is a surjective homomorphism $p : H \to G$ such that the kernel of p is isomorphic to an algebraic vector group.

(4.1) The tangent bundle $s : T(G) \to G$ and the prolongation $\pi : \tau(G) \to G$ are extensions of G by vector groups.

Let A be an abelian variety of dimension d. We will need one important property of extensions of abelian varieties by vector groups (see [R]).

(4.2) **Theorem** There is $p : \widehat{A} \to A$ an extension of A by a vector groups such that if $i : B \to A$ is an extension of A by a vector group, then there is a unique $j : \widehat{A} \to B$ such that $p = i \circ j$. We call $\widehat{A}$ the *universal extension of A by a vector group.* The dimension of $\widehat{A}$ is $2d$.

Let $\pi : \tau(\widehat{A}) \to \widehat{A}$ be the prolongation of A.

(4.3) There is $j : \widehat{A} \to \tau(\widehat{A})$ a section of π.

Since $p \circ \pi : \tau(\widehat{A}) \to A$ is an extension of A, by a vector group there is $j : \widehat{A} \to \tau(\widehat{A})$ such that $p = p \circ \pi \circ j$. Since $\pi \circ j : \widehat{A} \to \widehat{A}$ and $p = p \circ (\pi \circ j)$ and $p = p \circ \mathrm{id}$, by (4.2) $\pi \circ j = \mathrm{id}$. Thus j is a section of π.

(4.4) If B is a proper algebraic subgroup of $\widehat{A}$, then $p(B)$ is a proper subgroup of A.

Suppose not. Since $\ker p|B$ is a vector subgroup of $\ker p$, there is a vector group $V \subseteq \ker p$ such that $\ker p = V \oplus \ker p|B$. For $a \in \widehat{A}$ choose $b \in B$ such that $p(a) = p(b)$. There is $x \in \ker p|B$ and $v \in V$ such that $a - b = x + v$. Then $a = (b - x) + v$. Hence $\widehat{A} = B \oplus V$. Since B is a proper subgroup $V \neq \emptyset$.

Since $p|B : B \to A$, there is $j : \widehat{A} \to B$ such that $p \circ j = p$. Let σ be an automorphism of $\widehat{A}$ fixing B and acting as an automorphism of V. Then $p|B \circ (j \circ \sigma) = p$, contradicting the universal property of $\widehat{A}$.

Define $\phi : \tau(\widehat{A}) \to \ker \pi$ by $\phi(x) = x - j(\pi(x))$ and $\psi : \widehat{A} \to \ker \pi$ by $\psi = \phi \circ \nabla$. Note that

$$\ker \psi = \{a \in \widehat{A} : \nabla(a) = j(a)\}.$$

(4.5) $\ker \psi$ is Zariski dense in $\widehat{A}$.

Let $U \subseteq \widehat{A}$ be Zariski open and let V be the range of $j|U$. Since V projects onto U, by (3.4) there is $a \in U$ such that $\nabla(a) \in V$. But then $\psi(a) = 0$.

In 4.7 we give a different proof of this.

(4.6) Suppose $p(\alpha) = a$ and $nb = a$ where $a, b \in A$ and $\alpha \in \widehat{A}$. There is $\beta \in \widehat{A}$ such that $n\beta = \alpha$. In particular since $\mathrm{Tor}(A)$ is infinite torsion so $\mathrm{Tor}(\widehat{A})$, where $\mathrm{Tor}(G)$ is the group of torsion elements of G.

Choose $\gamma \in \widehat{A}$ such that $p(\gamma) = b$. Since $p(n\gamma) = a$, $\alpha - n\gamma \in \ker p$, a vector group. Thus there is $\beta \in \ker p$ with $n\beta = \alpha - n\gamma$. Then $n(\beta + \gamma) = \alpha$ and $p(\beta + \gamma) = b$.

(4.7) Suppose B is the Zariski closure of the torsion of $\widehat{A}$. Since $p(B)$ contains the torsion of A, $p(B) = A$. By 4.4, $B = \widehat{A}$. Since $\psi(\widehat{A})$ is a vector group, $\ker \psi \supseteq \mathrm{Tor}(\widehat{A})$ and $\ker \psi$ is Zariski dense in $\widehat{A}$.

(4.8) $\ker \psi$ is connected.

By 4.6 $\widehat{A}$ is divisible. If $\psi(\alpha) = 0$ and $n\beta = \alpha$, then $0 = \psi(\alpha) = n\psi(\beta)$. Since $\ker \pi$ is a vector group, $\psi(\beta) = 0$. Hence $\ker \psi$ is divisible and has no finite index subgroups.

(4.9) The field of differential functions on $\psi(V)$ is exactly the field of rational functions on $\widehat{A}$.

Let α be a differentially generic point of $\ker\psi$. Since $\nabla(\alpha) = j(\alpha)$, for any differential rational function f over K there is a rational function g over K such that $f(\alpha) = g(\alpha)$.

(4.10) If V is a proper Kolchin closed subset of $\ker\psi$, then V is not Zariski dense in $\widehat{A}$. Thus $\ker\psi$ is a minimal Zariski dense Kolchin closed subset of $\widehat{A}$.

If not then there is a differential equation holding on V, but not on $\psi(\widehat{A})$. By 4.9 this will give rise to an algebraic equation holding on V, but not on $\widehat{A}$.

Let $H = \ker\pi/\psi(\ker p)$ and let $\mu : A \to H$ be the map induced by ψ.

(4.11) $\ker\mu = p(\ker\psi)$.

If $\psi(\alpha) = 0$, then $\mu(p(\alpha)) = 0$.

If $\mu(a) = 0$ and $p(\alpha) = a$, the $\psi(\alpha) \in \psi(\ker p)$ so there is $\beta \in \ker p$ with $\psi(\alpha) = \psi(\beta)$ Thus $\alpha - \beta \in \ker\psi$ and $p(\alpha - \beta) = a$.

Let $A^{\#}$ be the Kolchin closure of $\mathrm{Tor}(A)$. Since $\mathrm{Tor}(\widehat{A})) \subseteq p^{-1}(\widehat{A}^{\#}) \cap \ker\psi$, $p^{-1}(A^{\#}) \cap \ker\psi$ is Zariski dense in $\widehat{A}$. Thus $\ker\psi \subseteq p^{-1}(A^{\#})$. By (4.8) $\ker\mu = p(\ker\psi) \subseteq A^{\#}$.

Thus $\ker\mu = A^{\#}$.

(4.12) Let $K(A^{\#})$ be the field of δ-rational functions on $\widehat{A}$. Then

$$\mathrm{td}(K(A^{\#})) \leq \mathrm{td}(K\ker\psi)) = \dim\, \widehat{A} = 2\dim\, A.$$

.

(4.13) $A^{\#}$ is a minimal Zariski dense subgroup of A.

Suppose $\Sigma \subset \ker\mu$ is a subgroup and $\Gamma = p^{-1}(\Sigma) \cap \ker\psi \subset \widehat{A}$. Then $p(\Gamma) = \Sigma$. Since A is the Zariski closure of Σ, the image of the Zariski closure of Γ under p is A. Hence, by 4.4, $\widehat{A}$ is the Zariski closure of Γ. Thus Γ is $\ker\psi$.

(4.14) If A is an abelian variety defined over the constants k, then $A^{\#} = A(k)$.

Since $\mathrm{Tor}(A) \subset A(k)$, $A^{\#} \subseteq A(k)$. Any differential algebraic subgroup of $A(k)$ is $B(k)$ for some algebraic group defined over k. If $B \subset A$, then $\mathrm{Tor}(B) \subset \mathrm{Tor}(A)$, so $A^{\#} = A(k)$.

We conclude with some remarks on H.

It is easy to see that H is a k-vector space. [Since $\{a \in K : a\psi(\ker p) = \psi(\ker p)\}$ is a subgroup containing $\mathbf{Z}$ it must contain k by strong minimality, and the quotient of k-vector spaces is a k-vector space.] Indeed the following theorem from [C] shows that H is differentially isomorphic to an algebraic K-vector group.

(4.15) **Theorem** (Cassidy) If G is an algebraic vector group and $\rho : G \to H$ is a surjective differential algebraic group homomorphism, then H is differentially isomorphic to an algebraic vector group.

§5 Model theory of Manin kernels

Let A be an abelian variety of dimension d.

If X is Kolchin closed and the differential function field of X has finite transcendence degree e then, RU $X \leq$ RM $X \leq e$. Using 4.12 we conclude that RM $A^{\#} \leq 2d$. While the notions of U-rank, Morley rank and transcendence degree may differ, one is finite if and only if all are finite in which case we refer to X as *finite dimensional.*

The following corollary is useful in diophantine applications; allowing us to replace finitely generated groups by "small" differential algebraic groups.

(5.1) If Γ is a finitely generated subgroup of A, then there is a finite dimension differential algebraic subgroup G of A containing Γ. Indeed $G \supseteq \{a \in A : na \in \Gamma$ for some $n = 1, 2, \ldots\}$.

Let $\gamma_1, \ldots, \gamma_n$ generate Γ. Since H is a k-vector space, we can take $H_0 \subset H$ the k-vector space generated by $\mu(\gamma_1), \ldots, \mu(\gamma_n)$. Let $G = \mu^{-1}(H)$. Using standard facts about finite rank groups we see that RU $G =$ RU $A^{\#} +$ RU $H \leq 2d + n$. Thus G is finite dimensional.

If A is defined over the constants k, then, by 4.14, $A^{\#}$ is $A(k)$. By quantifier elimination and stability, all definable subsets of $A(k)$ are already definable in the field language. Thus in this case, or more generally if A is isomorphic to an abelian variety defined over k, $A^{\#}$ is intimately related to k. If A is not isomorphic to an abelian variety defined over the constants, we will see that $A^{\#}$ behaves much differently.

For the remainder of this section we will assume that A is simple, i.e. there are no proper abelian subvarieties of A.

(5.2) If A is simple, then $A^{\#}$ has no infinite definable proper subgroups.

If $\Sigma \subset A$ is an infinite subgroup, then, since A is simple, Σ is Zariski is dense contradicting 4.13.

Let $X \subseteq A^{\#}$ be strongly minimal containing the identity.

(5.3) For some natural number n, $A^{\#} = \underbrace{X \cdot X \cdots X}_{n-\text{times}}$. In particular $A^{\#}$ is interpretable in X.

Since $A^{\#}$ has no infinite definable subgroups, X is indecomposable (for every definable subgroup H of $A^{\#}$, X/H is either infinite or has cardinality 1). The claim follows from Zilber's Indecomposability Theorem.

We will investigate $A^{\#}$ by investigating the nature of the strongly minimal set X. Since there is an infinite group definable interpretable in X, X is non-trivial. There are two cases depending on whether or not X is locally modular. For the remainder of this section we will assume a passing familiarity with the geometry of strongly minimal sets (see [P1] for details or [Mr2] for a brief survey).

(5.4) If X is locally modular, then, since $A^{\#}$ is interpretable in X, $A^{\#}$ is 1-based. By Hrushovski-Pillay every definable $Y \subseteq (A^{\#})^n$ is a finite boolean combination of cosets of definable subgroups of $(A^{\#})^n$.

(5.5) If X is locally modular, then $A^{\#}$ is strongly minimal.

By 5.4, X is a finite boolean combination of definable subgroups of $A^{\#}$. But, by 5.2, any definable subgroup of $A^{\#}$ is finite. Thus $A^{\#}$ differs from X by a finite set and $A^{\#}$ is strongly minimal.

(5.6) If X is locally modular, then A is non-isomorphic to an abelian variety defined over k.

If A is isomorphic to an abelian variety B defined over k, then $A^{\#}$ is isomorphic to $B(k)$ the k-rational points of B which can not be 1-based.

Remarkably Hrushovski and Sokolovic showed that the dichotomy suggested by 5.6 is sharp. That is if X is non-locally modular then A is isomorphic to an abelian variety defined over k. We give sketch the proof of this result outlining the model theoretic machinery of the proof. For further details the reader should consult [HS], [Bo] and [P2]

The first step is investigating the induced Kolchin topology on X^n.

Suppose D is an infinite set and we have a family of Noetherian topologies on that $D, D^2, \ldots$. We say that D is a *Zariski geometry* if:

i) If $f : D^n \to D^m$ is given by $f(x) = (f_1(x), \ldots, f_m(x))$ where each f_i is either a constant or projection, then f is continuous.

ii) The diagonal $\Delta_{n,i} = \{(x, y) \in D^n \times D^n : x_i = y_i\}$ is closed.

iii) If $C \subset D^{n+1}$ is an irreducible closed set and $\pi : D^{n+1} \to D^n$ is a coordinate projection then there is a proper closed F contained in $\overline{\pi(C)}$, the closure of $\pi(C)$, such that $\pi(C) \supseteq \overline{\pi(C)} \setminus F$.

iv) D is irreducible and for any closed $C \subset D^n \times D$ if C_a is either D or finite for all $a \in D^n$ where $C_a = \{x \in D : (a, x) \in C\}$, then there is N such that either $C_a = D$ or $|C_a| \leq N$ for all $a \in D^n$.

v) If C_1 and $C_2 \subseteq D^n$ are irreducible closed sets and W is a non-empty irreducible component of $C_1 \cap C_2$, then

$$\dim W + n \geq \dim C_1 + \dim C_2$$

where $\dim C = \sup\{\dim A + 1 : A \text{ is a proper irreducible closed subset of } C\}$. This is an ordinal notion of dimension that makes sense in any Noetherian topological space.

Conditions i) and ii) give a coherence between the topologies on D and its products. Condition iii) is a weak form of quantifier elimination while iv) is a form of strong minimality. Condition v) tells us that we can control the dimension of intersections. The classical example of a Zariski geometry is obtained my taking the Zariski topologies on D^n where D is a smooth algebraic curve. The fundamental results on Zariski geometries are proved in [HZ2], surveys are presented in [HZ1] and [M3]. The later reference also contains a proof of the next result.

(5.7) **Theorem** (Hrushovski-Sokolovic) If D is a strongly minimal set in a differentially closed field, then there is a finite $D \subseteq X$ such that $D \setminus F$ is a Zariski geometry where $(D \setminus F)^n$ has the induced Kolchin topology.

The next result is the fundamental theorem of Zariski geometries. Unlike arbitrary non-locally modular strongly minimal sets in Zariski geometries non-locally modularity indicates the presence of an algebraically closed field.

(5.8) **Theorem** (Hrushovski-Zilber) If D is a non locally modular Zariski geometry, then there is an interpretable algebraically closed field F non-orthogonal to D.

The field F is necessarily of finite rank. Using results of Cassidy, Sokolovic (see [P2]) classified all such fields.

(5.9) If F is a finite rank field interpretable in K, then F is definably isomorphic to k.

Combining (5.7)-(5.9) we can conclude:

(5.10) If $X \subseteq A^{\#}$ is a non-locally modular strongly minimal set, then X is non-orthogonal to k.

We can now use the theory of finite rank groups (see [Po]). Since $A^{\#}$ is interpretable in X and X is non-orthogonal to k, we can use Hrushovski's work on non-orthogonality in groups to conclude that there is a group H interpretable in k and a surjective homomorphism $\phi : A^{\#} \to H$ with finite kernel.

(5.11) There is an algebraic group G defined over k and a surjective homomorphism $\phi : A^{\#} \to G(k)$, the k-rational points of G.

This follows since any group definable in k is definably isomorphic to the k-rational points of an algebraic group defined over k.

We can now prove the converse to 5.6.

(5.12) **Theorem** (Hrushovski-Sokolovic) If X is non-locally modular, then A is isomorphic to an abelian variety defined over k.

Let $\phi : A^{\#} \to G(k)$ be as in 5.11. Let $N = |\ker \phi|$. For $x \in G(k)$ choose $y \in A^{\#}$ such that $\phi(y) = x$. If $\phi(y_1) = x$, then $y - y_1 \in \ker \phi$ and $Ny = Ny_1$. Thus we can define a homomorphism $\psi : G(k) \to A^{\#}$ by $\psi(x) = Ny$. Since A has only finitely many N-torsion points, $\ker \psi$ is finite. Since $A^{\#}$ has no proper definable infinite subgroups, ψ is surjective.

A priori ψ is a differential algebraic map, but since the domain is a set of points with constant coordinates the derivation adds nothing and we may assume that ψ is rational. Since $G(k)$ is Zariski dense in $G(K)$ and $A^{\#}$ is Zariski dense in $A(K)$, ψ is a surjective algebraic group homomorphism from G to A. Let L be the maximal linear algebraic subgroup of G. By a theorem of Chevalley (see [S]), G/L is an abelian variety. Since there are no non-constant algebraic maps from L to A, $L \subseteq \ker \psi$. Thus we may mod out by L. Since L is defined over k, we may, without loss of generality, assume that G is an abelian variety defined over k. In this case $\ker\psi$ is an abelian subvariety of G, which, by the rigidity of abelian varieties (see [Mi]) is defined over k. Thus $G/\ker\psi$ is an abelian variety defined over k isomorphic to A.

Let us quickly describe one easy diophantine consequence of these ideas. To see the details of Hrushovski's proof of the Mordell-Lang conjecture see [H], [P3], or [Bo].

(5.13) Let A be a simple abelian variety defined over $\mathbf{C}$ that is not isomorphic to an abelian variety defined over a number field. If $V \subset A$ is a proper subvariety of A, then V contains only finitely many torsion points of A. Indeed $A^{\#} \cap V$ is finite.

We can define a derivation δ on $\mathbf{C}$ such that the constant field k is the algebraic closure of $\mathbf{Q}$. We can extend δ to a differentially closed $K \supseteq \mathbf{C}$ with the same field of constants (indeed we could define δ on $\mathbf{C}$ making it a differentially closed field). Since A is not isomorphic to an abelian variety defined over k, $A^{\#}$ is strongly minimal. Thus $A^{\#} \cap X$ is either finite or $A^{\#}$. If $X \supset A^{\#}$ then X is Zariski dense, a contradiction.

In conclusion we remark that all non-trivial locally modular strongly minimal sets in differentially closed fields arise this way.

(5.14) If $X \subset K^n$ is a non-trivial locally modular strongly minimal set then, there is a simple abelian variety A non-isomorphic to any abelian variety defined over k, such that X is non-orthogonal to $A^{\#}$. Moreover for A and B simple varieties non-isomorphic to any abelian varieties defined over k, then $A^{\#}$ is non-orthogonal to $B^{\#}$ if and only if A and B are isogenous.

See [HS] or [P2].

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