The Borel Complexity of Ismomorphism for Theories with Many Types *

David Marker
University of Illinois at Chicago
marker@math.uic.edu

During the Notre Dame workshop on Vaught's Conjecture, Hjorth and Kechris asked which Borel equivalence relations can arise as the isomorphism relation for models of a first order theory. In particular, they asked if the isomorphism relation can be essentially countable but not tame. We show this is not possible if the theory has uncountably many types.

I am grateful to the logic group at Notre Dame for organizing this stimulating workhop.

1 Preliminaries

We begin by recalling the basic defintions and background material.

Suppose E_i is an equivalence relation on a standard Borel space X_i for $i = 1, 2$.

We say that E_1 is *Borel reducible* to E_2 if there is a Borel measurable $f : X_1 \to X_2$ such that xE_1y if and only if $f(x)E_2f(y)$ for all $x, y \in X_1$.

An equivalence relation is *countable* if every equivalence class is countable and *essentially countable* if it is Borel reducible to a countable equivalence relation. If $E_1 \leq_B E_2$ and $E_2 \leq_B E_1$, we write $E_1 \approx_B E_2$. A Borel equivalence relation E on X is *tame* if there is a Polish space Y and a Borel measurable $f : X \to Y$ such that xEy if and only if $f(x) = f(y)$.

If $\mathcal{L}$ is a countable first order language we let $X_{\mathcal{L}}$ be the Polish space of $\mathcal{L}$-structures with universe $\mathbb{N}$. For $\sigma \in \mathcal{L}_{\omega_1,\omega}$ let $\mathrm{Mod}(\sigma)$ be the Borel set of

*Partially supported by NSF grant DMS-0200393.

$\mathcal{M} \in X_{\mathcal{L}}$ with $\mathcal{M} \models \sigma$ and let $\cong_\sigma$ be the equivalence relation of isomorphism on Mod(σ). In general, $\cong_\sigma$ is $\mathbf{\Sigma}^1_1$, but need not be Borel.

Theorem 1.1 *For each countable Borel equivalence relation E, there is $\sigma \in \mathcal{L}_{\omega_1,\omega}$ such that $E \approx_B \cong_\sigma$.*

We give a quick sketch of the proof.[1] Since E is a countable Borel equivalence relation, by the Feldman-Moore Theorem [2], E is the orbit equivalence relation of a Borel action of a countable discrete group G on a Polish space X. Let $E(G, 2^\omega)$ be the natural shift action of G on $(2^\omega)^G$. There is a Borel reduction of E to $E(G, 2^\omega)$, see [1] 1.2.

Let $\mathcal{L} = \{\widehat{g} : g \in G\} \cup \{U_n : n \in \omega\}$ where each $\widehat{g}$ is a unary function and U_n is a unary predicate. Let σ be an $\mathcal{L}$-sentence such that $\mathcal{M} \models \sigma$ is a principle homogeneous space for the action $\alpha(g, x) = \widehat{g}(x)$ of G on $\mathcal{M}$. For $\mathcal{M} \models \sigma$, we associate $f_{\mathcal{M}} \in (2^\omega)^G$ where $f_{\mathcal{M}}(g)(n) = 1$ if and only if $\mathcal{M} \models U_n(\widehat{g}(0))$. This is a Borel isomorphism between Mod(σ) and $(2^\omega)^G$ and $\mathcal{M} \cong \mathcal{N}$ if and only if $f_{\mathcal{M}} E(G, 2^\omega) f_{\mathcal{N}}$. It follows that $\cong_\sigma$ is a countable Borel equivalence relation and there is $f : X \to \text{Mod}(\sigma)$ a Borel reduction of E to $\cong_\sigma$.

Since E has countable classes the map f is countable-to-one. Thus $f(X)$ is Borel and there is a Borel measurable $g : f(X) \to X$ such that $f(g(x)) = x$.

Let $C = \{\mathcal{M} \models \sigma : \exists \mathcal{N} \in f(X) \mathcal{N} \cong \mathcal{M}\}$. Since $f(X)$ is Borel and $\cong_\sigma$ is a countable equivalence relation, C is Borel. Clearly C is $\cong$-invariant. But every $\cong$-invariant Borel set is Mod(τ) for some $\mathcal{L}_{\omega_1,\omega}$-sentence τ. Clearly f is a reduction of E to $\cong_\tau$. There is a Borel $h : \text{Mod}(\tau) \to f(X)$ such that $\mathcal{M} \cong h(\mathcal{M})$. Then $g \circ h$ is a reduction of $\cong_\tau$ to E. Thus $E \approx_B \cong_\tau$.

Hjorth and Kechris asked if the same result was true for first order theories. It is easy to give example of theories T with continuum many countable models where $\cong_T$ is tame. For example, let T be the theory of an equivalence relation with infinitely many classes where each class contains an algebraically closed field. Then models are determined up to isomorphism by the set of transcendence degrees of the equivalence classes. Are there any first order theories T with $\cong_T$ essentially countable but not tame? We show that any such theory must have few types.

Let $\mathcal{C}$ be the Cantor space 2^ω. Fix $\langle\ ,\ \rangle : \omega^2 \to \omega$ a pairing function. For $x \in \mathcal{C}$, let $A_x \subseteq \mathcal{C}$ be the set $\{x_1, x_2, \ldots\}$ where $x_i(j) = x(\langle i, j\rangle)$. We say $x E_{\text{cnt}}\, y$ if and only if $A_x = A_y$.

[1] I am grateful to Kechris for showing me this argument.

The equivalence relation $E_{\rm cnt}$ is not essentially countable. See [3] Exercise 2.64.

Theorem 1.2 *Let T be a first order theory in a countable language where the type space $S(T)$ is uncountable. Then $E_{\rm cnt} \le_B \cong_T$. Thus $\cong_T$ is not essentially countable.*

This result is not surprising as the set of realized types is a natural invariant of a model.

2 Theories with Many Models

Suppose T is a first order theory in a countable language with $S(T)$ uncountable.

We can find $\mathcal{T}$ a perfect tree of types in $S(T)$. Choose $r_T \in \mathcal{C}$ such that $\mathcal{L}, T, \mathcal{T} \le_T r_T$. Using $\mathcal{T}$ we can code elements of the Cantor space as types.

Lemma 2.1 *There is continuous one-to-one map $\tau : \mathcal{C} \to S(T)$ such that $\tau(x) \le_T x \oplus r_T$ and $x \le_T \tau(x) \oplus r_T$, where $x \oplus y$ is the join of x and y.*

Scott Sets

Definition 2.2 We say that $S \subseteq \mathcal{C}$ is a *Scott Set* if

i) if $x \in S$ and $y \le_T x$, then $x \in S$;

ii) if $x, y \in S$, then $x \oplus y \in S$;

iii) if $x \in S$ codes an infinite subtree t of $2^{<\omega}$, then there is $y \in S$ an infinite path through t.

We need a refinement of recursively saturated models.

Definition 2.3 Let T be a complete first order theory in a countable language and let S be a Scott set with $T \in S$. We say that $\mathcal{M} \models T$ is *S-saturated* if:

i) for all $x \in S$ if $a_1, \ldots, a_n \in M$ and $p(v, a_1, \ldots, a_n)$ is a partial type recursive in some $x \in S$, then p is realized in $\mathcal{M}$;

ii) $\mathrm{tp}(a_1, \ldots, a_n) \in S$ for all $a_1, \ldots, a_n \in M$.

S-saturated models were studied in papers of Knight and Nadel ([4],[5]) and Wilmers [7]. The next result summarizes the facts that we will need.

Proposition 2.4 *Let T be a first order theory in a countable language. Let S be a countable Scott set with $T \in S$.*

i) There is a countable S-saturated model of T.

ii) S-saturated models of T are ω-homogeneous.

iii) Any two countable S-saturated models of T are isomorphic.

The proof of i) is a Henkin arguement where one alternates trying to realize types in S, witnessing existential sentences and making sure that for all Henkin constants $c_1, \ldots, c_n$, $\mathrm{tp}(c_1, \ldots, c_n) \in S$. The uniformity of this construction (and the uniqueness of S-saturated models) allows us to prove the following.

Lemma 2.5 *Let $\mathcal{S} = \{x \in \mathcal{C} : A_x$ is a Scott set$\}$. Then $\mathcal{S}$ is Borel and there is a Borel $\mu : \mathcal{S} \to \mathrm{Mod}(T)$ such that $\mu(x)$ is the A_x-saturated model of T.*

In fact, by the main result of [6], if $T \in A_x$, then an A_x-saturated model can be constructed recursively in x.

Borel Closure Systems

Let $\mathcal{F} = \{f_1, f_2, \ldots\}$ be a countable set of Borel functions $f_i : \mathcal{C}^{m_i} \to \mathcal{C}$. For $A \subseteq \mathcal{C}$, let $\mathrm{cl}_{\mathcal{F}}(A)$ be the closure of A under the functions in $\mathcal{F}$.

Definition 2.6 We say that $I \subseteq \mathcal{C}$ is *$\mathcal{F}$-independent* if

$$\mathrm{cl}_{\mathcal{F}}(A) \cap I = A$$

for all $A \subseteq \mathcal{I}$.

Lemma 2.7 *For any countable set of Borel functions $\mathcal{F}$, there is a perfect $\mathcal{F}$-independent set.*

Proof If P is a perfect set of suitably generic Cohen reals, then P is $\mathcal{F}$-independent.

Let $\mathcal{F}$ be the following collection of functions:

i) $j(x, y) = x \oplus y$;

ii)

$$f_e(x) = \begin{cases} \phi_e^x & \text{if } \phi_e^x \text{ is a total function in } \mathcal{C} \\ x & \text{otherwise} \end{cases} \quad \text{for } e = 0, 1, \ldots.$$

iii) $t(x)$ = leftmost path in the tree coded by x if x codes a tree on $2^{<\omega}$ and $t(x) = x$ otherwise.

iv) the constant function $x \mapsto r_T$.

If $A \subseteq \mathcal{C}$, then $\mathrm{cl}_{\mathcal{F}}(A)$ is a Scott set containing $A \cup \{r_T\}$. The construction of closures is uniform.

Lemma 2.8 *There is a Borel $\nu : \mathcal{C} \to \mathcal{C}$ such that $A_{\nu(x)}$ is the $\mathcal{F}$-closure of A_x for all $x \in \mathcal{C}$. In particular, $A_{\nu(x)}$ is a Scott set containing $A_x \cup \{r_T\}$.*

Proof of Theorem 1.2

Let P be a perfect $\mathcal{F}$-independent set with $\rho : \mathcal{C} \to P$ a homeomorphism. There is a Borel $\rho^* : \mathcal{C} \to \mathcal{C}$ such that $A_{\rho^*(x)} = \rho(A_x)$.

For $A \subseteq \mathcal{C}$ countable, let $S_A = \mathrm{cl}_{\mathcal{F}}(\rho(A))$ and let $\mathcal{M}_A$ be the unique countable S_A-saturated model of T.

Lemma 2.9 *If $A \neq B$, then $\mathcal{M}_A \not\cong \mathcal{M}_B$.*

Proof Suppose $x \in A \setminus B$. Then $\rho(x) \in S_A$, but, since P is $\mathcal{F}$-independent, $\rho(x) \notin S_B$. Since $r_T \in S_A \cap S_B$, it follows from Lemma 2.1, that $\tau(\rho(x)) \in S(T) \cap S_A$ and $\tau(\rho(x)) \notin S(T) \cap S_B$. The type $\tau(\rho(x))$ is realized in $\mathcal{M}_A$ but not $\mathcal{M}_B$. Thus $\mathcal{M}_A \not\cong \mathcal{M}_B$.

We now build our reduction of E_{cnt} to $\cong_T$. For $x \in \mathcal{C}$, let $g(x) = \mu(\nu(\rho^*(x))$. Unravelling the definiton:

i) $A_{\rho^*(x)} = \rho(A_x)$;

ii) $A_{\nu(\rho^*(x))} = \mathrm{cl}_{\mathcal{F}}(\rho(A_x))$;

iii) $g(x)$ is a code for a $\mathrm{cl}_{\mathcal{F}}(\rho(A_x))$-saturated model of T.

Since S-saturated models are unique, if $x E_{\mathrm{cnt}}\, y$, then $g(x) \cong g(y)$. By Lemma 2.9, if $x \not{E}_{\mathrm{cnt}}\, y$, then $g(x) \not\cong g(y)$. Thus $E_{\mathrm{cnt}} \leq_B \cong_T$.

Remarks

Let $\mathrm{hMod}(T) \subseteq \mathrm{Mod}(T)$ be the codes for homogeneous models of T. Countable homogeneous models are determined by the pure types they realize.

Corollary 2.10 *Suppose $S(T)$ is uncountable, then $E_{\mathrm{cnt}} \approx_B \cong_T \mid \mathrm{hMod}(T)$.*

Problem Find a first order theory T where $\cong_T$ is not tame and $E_{\mathrm{cnt}} \not\leq_B \cong_T$. Note that counterexamples to Vaught's conjecture have this property.

Is there a ω-stable theory with this property?

References

[1] R. Dougherty, S. Jackson and A. Kechris, The structure of hyperfinite Borel equivalence relations, Trans. Amer. Math. Soc. 341 (1994), no. 1, 193–225.

[2] J. Feldman and C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras. I. Trans. Amer. Math. Soc. 234 (1977), no. 2, 289–324.

[3] G. Hjorth, *Classification and Orbit Equivalence Relations*, Amer. Math. Soc. Mathematical Surveys and Monographs vol 75, Providence, 2000.

[4] J. Knight and M. Nadel, Expansions of models and Turing degrees. J. Symbolic Logic 47 (1982), no. 3, 587–604.

[5] J. Knight and M. Nadel, Models of arithmetic and closed ideals. J. Symbolic Logic 47 (1982), no. 4, 833–840.

[6] A. Macintyre and D. Marker, Degrees of recursively saturated models. Trans. Amer. Math. Soc. 282 (1984), no. 2, 539–554.

[7] G. Wilmers, Minimally saturated models. *Model theory of algebra and arithmetic*, pp. 370–380, Lecture Notes in Math., 834, Springer, Berlin, 1980.