DEGREES CODING MODELS OF ARITHMETIC

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INTRODUCTION

In 1959 Tennenbaum proved that if $M$ is a nonstandard model of Peano arithmetic, (PA) then the addition and multiplication of $M$ can not be recursive. This result leads to several natural questions:

i) Given $T$ a complete extension of Peano arithmetic what Turing degrees code models of $T$?

ii) Given a model $M$ of $T$, what Turing degrees code $M$?

This thesis begins with a survey of known results in this area including proofs of many folklore results. It also represents my contributions to the study of these problems.

In Section 1 we introduce Scott sets, the basic combinatorial tool used in studying degrees coding models of arithmetic, and begin to study the relationships between a model, the degrees that code it, its reals and its complete theory. Of the results in this section only 1.21 and 1.27 are new.

Section 2 is devoted to classification of degrees coding complete extensions of PA.

In Section 3 we consider what degrees code nonstandard models of $\text{Th}(\mathbb{N})$. We survey the results which led up to Solovay's classification theorem.

In Section 4 we give the proof of Theorem 3.5, which states that if every arithmetic set is recursive in $d$ then there is a nonstandard model of $\text{Th}(\mathbb{N})$ recursive in $d'$. This is the main theorem of [M].

In Section 5 we give a classification of degrees coding S-saturated structures. We also review the necessary model theoretic preliminaries.
In Section 6 we apply the results of Section 5 to the study of degrees coding certain reducts and definable substructures of models of arithmetic.

In Section 7 we classify the degrees coding reducts and definable substructures of a fixed completion of \( \text{PA} \). We also give an improvement of Solovay's classification of degrees coding models of \( \text{Th}(\mathbb{N}) \) and an application. Many of the results of Sections 5, 6 and 7 are joint work with Angus Macintyre and will appear in [Mac-M].

In Section 8 we study the degrees coding nonstandard models of a theory introduced to study the polynomial time hierarchy. We show that any such degree must be above \( \mathcal{Q}' \).

Our model theoretic notation is standard. From recursion theory: \( \{e\}^X \) denotes the \( e \)th partial recursive function with oracle \( X \), \( W^X_e \) denotes the domain of \( \{e\}^X \), and if \( d \subseteq \omega \), \( \mathcal{d} \) denotes the Turing degree of \( d \). (Every Turing degree will be thought of as \( \mathcal{d} \) where \( d \subseteq \omega \) is some canonically chosen representative).
§1. SCOTT SETS

In this section we will lay the groundwork for our study of degrees coding models of arithmetic by examining the relationship between models of arithmetic and their associated Scott sets.

Throughout $2^\omega$ will denote finite sequences of zeros and ones. There is a natural recursive coding of elements of $2^\omega$ as elements $\omega$. When no difficulties arise we will shamelessly confuse $2^\omega$, $\omega$, and $\omega^n$ when it is convenient to do so. Similarly we confuse $P(\omega)$ and $2^\omega$.

Definition 1.1: $T \subseteq 2^\omega$ is a tree iff for any $\sigma \in T$ if $\tau \subseteq \sigma$, then $\tau \in T$. (Conforming to the above convention we will speak of $X \subseteq \omega$ as a tree if the subset of $2^\omega$ which codes it is a tree.)

Definition 1.2: $S \subseteq P(\omega)$ is a Scott set iff

i) $x_1, \ldots, x_n \in S$ and $Y \preceq_{T} X_1 \times X_2 \times \cdots \times X_n$, then $Y \in S$.

ii) If $T \in S$ is an infinite tree, then there is $f \in S$ an infinite path through $T$.

The next proposition gives us the most usefull property of Scott sets for model theorists. Let $L$ be some fixed recursive language. We will identify $L$-sentence (or $L$-formulas) with elements of $\omega$ via Gödel codes.

Proposition 1.3: Let $S$ be a Scott set. If $T \in S$ is an $L$-theory, then $T$ has a completion $T^* \in S$.

Proof: Let $\varphi_1, \varphi_2, \ldots$ be a listing of all $L$-sentences. Let $X = \{ \sigma \in 2^\omega$: for any $\tau \subseteq \sigma$, there is no proof of $\begin{aligned} &7 (\bigwedge_{i=0}^{\tau(i)=1} \varphi_i) \bigwedge &\bigwedge_{i=1}^{\tau(i)=0} \varphi_i \end{aligned}$ from $T$ with Gödel code at most $|\sigma|$ $(|\sigma|$ denotes the length of $\sigma$). $X$ is recursive in $T$ so $X \in S$. As $S$ is a Scott
set there is an infinite path \( f \in 2^\omega \cap S \). Clearly \( T^* = \{ \varphi_i : f(i) = 1 \} \) is a completion of \( T \).

Scott sets arise naturally when studying nonstandard models of Peano arithmetic (PA).

**Definition 1.4:** Let \( M \models PA \). If \( a \in M \) we define the real coded by \( a \),

\[
 r(a) = \{ n \in \omega : M \models p_n \upharpoonright a \},
\]

where \( p_n \) denotes the \( n \)th prime number. We let \( \Re(M) \) denote \( \{ r(a) : a \in M \} \).

**Proposition 1.5:** (Scott-Tennenbaum [Sc-T]). If \( M \models PA \) is nonstandard, then \( \Re(M) \) is a Scott set.

**Proof:** Let \( d \in M \) be some nonstandard element.

**Claim 1:** Suppose \( a, b \in M \). Let \( \sigma : M \to M^2 \) be the usual recursive pairing function \( a \mapsto \langle \sigma_0(a), \sigma_1(a) \rangle \). As \( M \) is a model of arithmetic \( M \models \forall a \forall w \forall i < d (p_i \upharpoonright w \leftrightarrow (p_{\sigma_0(i)} \upharpoonright u \land p_{\sigma_1(i)} \upharpoonright v)) \). Thus there is \( c \in M \) such that for any \( n \in \omega \), \( M \models p_n \upharpoonright c \leftrightarrow (p_{\sigma_0(n)} \upharpoonright a \land p_{\sigma_1(n)} \upharpoonright b) \). Thus \( r(c) \) codes \( r(a) \times r(b) \).

**Claim 2:** Suppose \( a \in M \) and \( A \preceq_T r(a) \). Say \( \{ e \}_{r(a)} \) is the characteristic function of \( A \). Let \( B = \{ b \in M : M \models p_b \upharpoonright a \} \). \( M \) believes \( B \) is finite. In \( M \) we can run computations \( \{ e \}_{B} \) for \( d \) steps. If we input a standard number \( n \), then \( \{ e \}_{B}(n) \) will only make oracle queries to \( B \) about standard numbers and \( B \) will give affirmative answers iff \( p_n \upharpoonright a \). So the computation \( \{ e \}_{r(a)}(n) \) will be identical to the computation \( \{ e \}_{B}(n) \). In particular, as \( \{ e \}_{B}(n) \) will halt at some standard stage, \( n \in A \) iff \( \{ e \}_{B}(n) \) converges in \( d \) steps and outputs one. As \( C = \{ m \in M : m \leq d \) and \( \{ e \}_{B}(m) \) halts in \( d \) steps and outputs 1 \} is \( M \)-finite, there is \( b \in M \) such that for \( n \leq d \) \( p_n \upharpoonright b \) iff \( n \in C \). Clearly \( A = r(b) \).
Claim 3: Suppose \( r(a) \) is an infinite tree \( T \). Consider 
\[ T' = \{ \sigma \in 2^\omega : p_\sigma | a \}. \]
For every standard \( n \), there is \( \sigma \in T' \) such that 
\[ |\sigma| = n \] and if \( \tau \subseteq \sigma \), then \( \tau \in T' \). By overspill there must be a 
nonstandard \( c \) and \( a \cdot c \in 2^\omega \) such that 
\[ |\sigma| = c, \sigma \in T' \] and every sub-
sequence of \( \sigma \) is in \( T' \). Every standard initial subsequence of \( \sigma \) is 
in \( 2^\omega \) so it must be in \( T \). Thus the standard initial subsequences of 
\( \sigma \) form an infinite path through \( T \). In \( M \) we can find a \( b \) such that 
for any sequence \( \tau \in 2^c \), \( p_\tau | b \) iff \( \tau \subseteq \sigma \). Clearly \( r(b) \) is a path 
through \( T \).

Proposition 1.5 is the key model theoretic step toward Tennenbaum's 
result that no nonstandard model of arithmetic can be recursively present-
ed. The next lemma is the key recursion theoretic step. Recall that 
two sets \( A \) and \( B \) are \( d \)-recursively inseparable iff \( A \cap B = \emptyset \) and 
there is no \( d \)-recursive \( X \) such that \( A \subseteq X \) and \( B \cap X = \emptyset \). Given any 
set \( d \) the sets \( A_d = \{ n : \{ n \}^d(n) \text{ converges with output } 1 \} \) and 
\( B_d = \{ n : \{ n \}^d(n) \text{ converges with output } 0 \} \) are \( d \)-recursively inseparable 
\( d \)-r.e.. See Rogers [R] for details.

Lemma 1.6: If \( S \) is a Scott set and \( d \in S \), then there is \( X \in S \) 
such that \( X \not\models_T d \).

Proof: Let \( T \) be the tree \( \{ \sigma \in 2^\omega : \text{If } n \in \text{dom } \sigma \text{ and } \{ n \}^d(n) \text{ converges within } |\sigma| \text{ steps, then } \{ n \}^d(n) = \sigma(n) \} \). Let \( f \in S \) be an 
infinite path through \( T \). Let \( X = \{ n : f(n) = 1 \} \). Clearly \( X \leq_T f \), so 
\( X \in S \). As \( X \) separates \( A_d \) and \( B_d \), \( X \not\models_T d \).

In particular, every Scott set contains a nonrecursive element.

Definition 1.7: If \( M \) is a countable model of arithmetic we can think 
of \( M \) as \( (\omega, \emptyset, \emptyset) \). We define \( \text{Diag}(M) \) to be the set of atomic sentences
true in $M$. We will let $\deg(M)$ denote the Turing degree of $\text{Diag}(M)$. Henceforth all models of arithmetic will be assumed countable and of the form $(\omega, \emptyset, \emptyset)$. It is useful to note that $\deg(M)$ is not an isomorphism invariant of $M$.

**Lemma 1.8:** If $M \models \text{PA}$ is nonstandard and $a \in M$, then $r(a) \leq \text{Diag}(M)$.

**Proof:** First note that $n \in r(a)$ iff $\exists y \underbrace{y + \cdots + y}_{p_n \text{-times}} = a$ so $r(a)$ is r.e. in $\text{Diag}(M)$. Further $n \notin r(a)$ iff

$$w \exists y \underbrace{y + y + \cdots + y}_{p_n \text{-times}} = a.$$ 

Thus $\{n : n \notin r(a)\}$ is r.e. in $\text{Diag}(M)$. Thus $r(a) \leq \text{Diag}(M)$. As $\Re(M)$ is a Scott set, by 1.6 $r(a) \leq \text{Diag}(M)$. //

For future use we note that the way of choosing an index for $r(a)$ in $\text{Diag}(M)$ is uniform in $a$. Tennenbaum's theorem is now obvious.

**Corollary 1.9:** (Tennenbaum [T]). If $M \models \text{PA}$ is nonstandard, then $\deg(M) \neq \emptyset$.

We do not need the full force of $\text{PA}$ to prove Corollary 1.9. McAlloon [Mc] shows that 1.9 still holds if we weaken the induction schema to formulas with bounded quantifiers. Wilmers [W3] has strengthened this by showing we need only the induction schema for formulas which in prenex form have only bounded existential quantifiers. On the other hand Shepherdson [Sh] has shown that if we restrict the induction schema to quantifier free formulas then we can find recursive nonstandard models.

After 1.8 we stated that $r(a)$ is recursive in $\text{Diag}(M)$ uniformly in $a$. Let us make this more precise.
Definition 1.10: Let $S \subseteq P(\omega)$ such that $|S| = \aleph_0$. Let $E \subseteq \omega^2$. $E$ is a $d$-enumeration of $S$ iff $E \equiv_T d$ and $S = \{E_n : n \in \omega\}$, where $E_n = \{m : (n, m) \in E\}$.

It might seem more natural to only require $E \subseteq_T d$, but the next lemma shows these notions are equivalent for all interesting $S$.

Proposition 1.11: If $S \subseteq P(\omega)$, $|S| = \aleph_0^3$, $\{X \in S : 0 \in X\}$ is infinite and co-infinite $e_T \geq d$ and $S$ has a $d$-enumeration $E$, then $S$ has an $e$-enumeration $E^*$.

Proof: Find $h : \omega \to \omega$ bijective such that $0 \in E_{h(n)}$ if $n \in e$ and $h \leq_T d$. Let $(n, m) \in E^*$ if $(h^{-1}(n), m) \in E$. Clearly $E^*$ is an enumeration of $S$. $E^* \leq_T e$ and since $n \in e$ if $0 \in E_{E_n}^*$, $e \leq_T d$. //

Our notion of enumeration is tied in with the notion of weak uniform upper bound in recursion theory. See Moser [Mo] for details.

Enumerations arise in arithmetic in the following way.

Lemma 1.12: If $\mathcal{M} \models PA$ is nonstandard, there is a $\text{Diag}(\mathcal{M})$-enumeration of $\beta(\mathcal{M})$.

Proof: Let $E \subseteq \omega^2$ be $\{(a, n) : \mathcal{M} \models p_n \upharpoonright a\}$. The proof of 1.8 and the subsequent remarks show that $E$ is an enumeration of $S$ and $E \leq_T \text{Diag}(\mathcal{M})$. The result follows from 1.11. //

The following notion of Solovay's seems unnatural but is quite useful.

Definition 1.13: If $S$ is a countable Scott set, a $d$-effective enumeration of $S$ is a collection $E \subseteq \omega^2$, $f_1 : \omega^2 \to \omega$, $f_2 : \omega^2 \to \omega$ and $f_3 : \omega \to \omega$ such that:
i) $E \equiv T^d$ and $f_1, f_2, f_3$ are $d$-recursive.

ii) $S = \{E_n : n \in \omega\}.$

iii) $E_{f_1}(m, n) = E_m \times E_n.$

iv) If $\{e\}^n_n \in 2^\omega$ then $E_{f_2}(e, n) = W_e^n.$

v) If $E_n$ is an infinite tree, $E_{f_3}(n)$ is an infinite path through $T.$

Surprisingly effective enumerations arise naturally.

Lemma 1.14: (Solovay [So]). If $M \models PA$ is nonstandard, then there is a $Diag(M)$-effective enumeration of $Re(M).$

Proof: Let $E$ be as in the proof of 1.12 (we will eventually have to use 1.11 to pad $E,$ but we ignore this complication). We must define $f_1, f_2$ and $f_3.$

1) In Claim 1 of the proof of 1.5 we showed that $M \models \forall u, v \exists w \exists i < d (p_{\sigma_1}(i) \vdash u \land p_{\sigma_1}(i) \vdash v).$ By Martin-\'{a}sevic's theorem [D-M-R], there is a quantifier free formula $\psi(u, v, w, x, z)$ such that $PA \models \forall i < x (p_{\sigma_1}(i) \vdash u \land p_{\sigma_1}(i) \vdash v)) \lnot \vdash \exists z \psi(u, v, w, x, z).$

We define $f_1$ as follows. Given $a, b$ we search for a $c$ and a $\bar{c}$ such that $\psi(a, b, c, d, \bar{c}).$ This search can be carried out effectively in $Diag(M).$ We will eventually find such a $c$ and $\bar{c}.$ The first time we do we let $f_1(a, b) = c.$

2) In Claim 2 of the proof of 1.5 we showed that $M \models \forall u, v \exists w \forall i < d (p_{\sigma_1}(i) \vdash \forall v \{b : Pb \mid u\}(i) \text{ converges in } d \text{ steps and outputs } 1),$ where this is suitably formalized using the $\Delta_0$ definition of the Kleene T-predicate. As above we can replace the bounded quantifier portion of this by an existential formula. We can effectively search for instantiations of the existential formulas and if $\{e\}^r(a) \in 2^\omega,$
the witnesses we find will give a c coding \( \mathcal{W}^r_e(a) \). Let \( f_2(e,a) = c \).

3) Given any \( u \in M, M \models "\exists v, \sigma \in 2^M \text{ such that } \sigma \text{ is in the finite subset of } 2^M \text{ coded by } \{b \in M: p_b | u\}, \text{ and } \sigma \text{ is of maximal length such that every initial subsequence of } \sigma \text{ is in the set coded by } \{b \in M: p_b | u\}, \text{ and } \forall i < |\sigma| p_i | v \iff \sigma(i) = 1." \) If \( r(a) \) is a tree and \( c \) witnesses this, then \( r(c) \) is an infinite path through \( T \). As in 1) and 2) above we can replace the \( \Delta_0 \)-part of this by an existential formula and effectively search for witnesses. We define \( f_3(a) \) to be the least such witness. //

At first \( d \)-effective enumerations seem much rarer than \( d \)-enumerations, but we will show in §7 that if there is a \( d \)-enumeration of \( S \), then there is a \( d \)-effective enumeration. From time to time we will tacitly use the following fact.

**Proposition 1.15:** If there is a \( d \)-effective enumeration of \( S \) and \( e \geq d \), then there is an \( e \)-effective enumeration.

**Proof:** As in 1.11. //

When studying models of arithmetic from a recursion theoretic point of view, we are forced to look for invariants of \( M \) which have a recursion theoretic character. One candidate is \( \text{Re}(M) \). Another is \( \text{Th}(M) \), the complete first order theory of \( M \). The remainder of this section is devoted to the relationship between \( \text{Re}(M) \), \( \text{Th}(M) \) and \( \text{deg}(M) \). Our principle model theoretic tool is the following result implicit in Friedman [Fr].

**Lemma 1.16:** Let \( M \) be a nonstandard model of \( \text{PA} \). Let \( \Gamma(v, \bar{c}) \) be a recursive set of \( \Sigma_n \)-formulas in the free variable \( v \) and parameters \( \bar{c} \), which is consistent with \( \text{Th}(M, \bar{c}) \). There is an \( a \in M \) realizing
\[ \Gamma(v, \bar{c}). \]

**Proof:** Let \( \text{Sat}_n \) be the \( \Sigma_n \)-predicate defining satisfaction for \( \Sigma_n \) formulas (i.e., if \( \varphi(\bar{v}) \) is a \( \Sigma_n \)-formula and \( \bar{a} \in M \), \( M \models \text{Sat}_n (\varphi(\bar{v}) \tau a \tau) \iff \varphi(\bar{a}) \)). Let \( \gamma_0, \gamma_1, \gamma_2, \ldots \) be a recursive listing of \( \Gamma \). Using the fact that recursive sets can be represented in arithmetic, we can find a formula \( \Gamma(v, w) \) such that for any \( a \in M \), 

\[ M \models \forall w(\psi(a, w) \iff \text{Sat}_n (\ \wedge \\gamma_i (v), w)). \] 

As \( \Gamma(v) \) is consistent, for any \( n \in \omega, M \models \exists w \psi(n, w) \). Thus, by overspill this must also be true for some nonstandard \( a \in M \). Let \( b \) witness this. Then \( b \) realizes \( \Gamma(v) \).

**Corollary 1.17:** If \( M \models \text{PA} \) is nonstandard, then for each \( n \in \omega \)

\[ \text{Th}(M) \cap \Sigma_n \in \text{Re}(M). \]

**Proof:** Let \( \Gamma(v) \) be the type \( \{ p_\varphi \mid v \iff \varphi : \varphi \in \Sigma_n \} \). By 1.16 \( \Gamma(v) \) is realized in \( M \).

**Corollary 1.18:** (Feferman [F]). If \( M \models \text{Th}(\mathbb{N}) \) is nonstandard, then \( \text{Re}(M) \) contains all arithmetic sets.

Knight showed that 1.17 gives the only restriction the complete theory of a countable model of \( \text{PA} \) with a given Scott set.

**Theorem 1.19:** (Knight [K1]). If \( S \) is a countable Scott set and \( T \) is a complete extension of \( \text{PA} \) such that every \( T \cap \Sigma_n \in S \), then there is \( M \models T \) with \( \text{Re}(M) = S \).

We will show that for any countable Scott set \( S \), there are arbitrarily complex \( T \) such that for any \( n \), \( T \cap \Sigma_n \in S \). We will need the following lemma used by Harrington [H].
Lemma 1.20: There is a recursive sequence of sentences \( \sigma_2, \sigma_3, \sigma_4, \ldots \)
such that \( \sigma_n \) is a \( \Delta_n \) sentence and \( \sigma_n \) is independent of \( \text{PA} \cup T \),
where \( T \) is any set of \( \Sigma_{n-1} \) sentences consistent with \( \text{PA} \).

Proof: Using Gödel's diagonalization lemma [Sml], we can find
a Rosser sentence \( \sigma_n \) such that
\[
\sigma_n \leftrightarrow "\text{If there is a proof of } \sigma_n \text{ from true } \Sigma_{n-1}\text{ sentences, then there is a proof with smaller Gödel code of } \bar{\sigma}_n \text{ from true } \Sigma_{n-1}\text{ sentences.}""
\]
Here by proof we always mean proof from \( \text{PA} \).

\( \sigma_n \) is easily seen to be \( \Pi_n \). Let \( \sigma'_n \) be the sentence "Either there is no proof of \( \sigma_n \) from true \( \Sigma_{n-1} \) sentences or there is a proof of \( \neg \sigma_n \) from true \( \Sigma_{n-1} \) sentences for which there is no proof of \( \sigma_n \) from true \( \Sigma_{n-1} \) sentences with a smaller Gödel code." The first disjunct of \( \sigma'_n \) is \( \Pi_{n-1} \), the second is \( \Sigma_n \). Thus \( \sigma_n \) is \( \Sigma_n \).

Further, \( \text{PA} \vdash \sigma_n \leftrightarrow \sigma'_n \), so \( \sigma_n \) is \( \Delta_n \).

Suppose \( T \subseteq \Sigma_{n-1} \) is consistent with \( \text{PA} \) and \( T \cup \text{PA} \vdash \sigma_n \).
Then there is a proof of \( \text{PA} \) from true \( \Sigma_{n-1} \) sentences, so there is a proof of \( \neg \sigma_n \). Hence \( \text{PA} \cup T \vdash \neg \sigma_n \), a contradiction. On the other hand if \( T \cup \text{PA} \vdash \neg \sigma_n \), then there is a proof of \( \sigma_n \) from true \( \Sigma_{n-1} \) sentences. Hence \( T \cup \text{PA} \vdash \sigma_n \), a contradiction. Thus \( \sigma_n \) is sufficiently independent.

Theorem 1.21: If \( S \) is a countable Scott set and \( d \leq \omega \), there is a complete extension \( T \supseteq \text{PA} \) such that \( T \cap \Sigma_n \in S \) for each \( n \in \omega \) and \( d \leq \Sigma_T \).

Proof: We will build \( T \) as a union of \( T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots \) such that for \( n \geq 1 \) \( T \cap \Sigma_n \subseteq T_n \subseteq \Sigma_{2n+2} \), each \( T_n \in S \) and \( T \) is a complete extension of \( \text{PA} \). Let \( T_0 = \emptyset \). Let \( \sigma_2, \sigma_3, \sigma_4, \ldots \) be as in Lemma 1.20.
Step n: Suppose we have $T_{n-1} \subseteq S_n$. Let $T^* = T_{n-1} \cup PA \cup \{\varphi\}$. Where

- $\varphi$ is $\sigma_{2n+1}$ if $n \in d$ and $\neg \sigma_{2n+1}$ if $n \notin d$. $T^*$ is consistent and
- $T^* \in S$. Thus by 1.3 there is a complete $T^{**} \supseteq T$ in $S$. Let $T_n = T^{**} \cap S_{2n+2}$. Clearly, $T_n \in S$.

Let $T = U T_n$. It is easily seen that each $T \cap S_n$ is in $S$ and as $n \in d$ iff $\sigma_{2n+1} \in T$, $d \leq T$.

In Theorem 1.27 we will show conditions sufficient to insure there is $T =_{T} d$ and in Section 7 we will show these conditions are necessary.

**Definition 1.22:** Let $T$ be a complete extension of $PA$. We say $X \subseteq \omega$ is represented in $T$ iff there is a parameter free formula $\varphi(v)$ such that for each $n \in \omega$ $n \in X$ iff $\varphi(n) \in T$. $\text{Rep}(T)$ is the set of all subsets of $\omega$ represented in $T$.

**Lemma 1.23:** If $T$ is a complete extension of $PA$, $T \neq \text{Th}(M)$ and $M$ is the minimal model of $T$, then $\text{Rep}(T) = \text{Re}(M)$.

**Proof:** First note that since $PA$ has definable Skolem functions, there is a pointwise definable minimal model of $T$. Let $\varphi(v)$ be a parameter free formula. Let $\Gamma(v) = \{p_n \mid v \leftrightarrow \varphi(n) : n \in \omega\}$. By 1.16 $\Gamma$ is realized in $M$ by some element $a$. But then $\varphi(n)$ iff $n \in r(a)$, so $\text{Rep}(T) \subseteq \text{Re}(M)$.

Let $a \in M$. As $M$ is minimal there is a formula $\psi(v)$ such that $M \models \psi(a) \land \forall v(\psi(v) \rightarrow w = a)$. Let $\varphi(v)$ be the formula $\forall w(\psi(w) \rightarrow p_v \mid w) \land \exists w(\psi(w) \land p_v \mid w)$. Then $p_n \mid a$ iff $\varphi(n) \in T$. So $r(a) \in \text{Rep}(T)$. Thus $\text{Rep} T = \text{Re}(M)$.

**Corollary 1.24:** (Scott-Tennenbaum [Sc-T]). If $T$ is a complete extension of $PA$, then $\text{Rep}(T)$ is a Scott set.
Proof: If $T \neq \text{Th}(\mathbb{N})$ this follows from 1.23. If $T = \text{Th}(\mathbb{N})$, then $\text{Rep}(T)$ is just the arithmetic sets. //

Scott [Sc] proved the converse of 1.24.

**Theorem 1.25:** (Scott [Sc]). If $S$ is a countable Scott set, there is a complete $T \supseteq \text{PA}$ such that $S = \text{Rep}(T)$.

We will modify Scott's proof to obtain an effective version of 1.25. We use the following lemma.

**Lemma 1.26:** For each $n \geq 2$ there is a recursive sequence of $\Delta_n$ sentences $\gamma_0^n, \gamma_1^n, \gamma_2^n, \ldots$ such that for any set of $\Sigma_{n-1}$ formulas $T$, which is consistent with $\text{PA}$, and any $\tau \in 2^{\omega} \cup T \cup \{ \wedge_{\tau(i)=1} \gamma_i^n \}$, $\tau \gamma_i^n$ is consistent.

**Proof:** Using Gödel's diagonalization lemma we can find $\gamma_i^n$ such that $\gamma_i^n \iff$ "For any proof of $\gamma_1^n$ from $\text{PA}$, true $\Sigma_{n-1}$ sentences and the true assignment of $\gamma_0^n, \ldots, \gamma_{i-1}^n$, there is a shorter proof of $\tau \gamma_i^n$ from the same hypothesis." The verification that this sequence works is similar to the verification in Lemma 1.20. We use heavily the inductive assumption that $\gamma_1^n, \ldots, \gamma_{i-1}^n$ are $\Delta_n$. //

**Theorem 1.27:** Suppose there is a $d$-effective enumeration of a Scott set $S$. Then there is a complete $T \supseteq \text{PA}$ such that $\text{Rep}(T) = S$ and $T = T_d$.

**Proof:** Let $E, f_1, f_2, f_3$ be the $d$-effective enumeration of $S$. Recursively in $d$, we build $T = \cup T_n$ where $T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$, $T_n \in S$ $T \cap \Sigma_n \subseteq T_n \subseteq \Sigma_{2n+2}$ and $T$ is a complete extension of $\text{PA}$. Let $T_0 = \emptyset$.

**Step $n = 2m$:** (We proceed as in the proof of 1.21). Let $s$ be such
that \( E_s = T_{n+1} \). Using \( f_1 \) and \( f_2 \) we can effectively find an \( s_0 \) such that \( E_{s_0} = T_{n+1} \cup PA \cup \{ \varphi \} \) where \( \varphi = \sigma_{2n+1} \) if \( n \in d \) and \( \sigma_{2n+1} \) if \( n \notin d \) (here \( \rho \in \omega \) is as in Lemma 1.20), and an \( s_1 \) such that \( E_{s_1} \) is a tree with every infinite path yielding a completion of \( E_{s_0} \). Using \( f_3 \) we can effectively find \( t \) so that \( E_t \) is a path through \( E_{s_1} \). Finally, using \( f_1 \) and \( f_2 \) we can find \( t' \) such that \( E_{t'} = E_t \cap \Sigma_{2n+2} \cdot \) Let \( T_n = E_{t'} \).

Step \( n = 2m+1 \): (We try to code \( E_m \) into \( T \cap \Sigma_{2n+1} \).) Let \( s \) code \( T_n \). Using \( f_1, f_2 \) and \( f_3 \) as above we can find a \( t \) so that \( T \) codes \( T^* \cap \Sigma_{2n+2} \cdot \) where \( T^* \) is a completion of \( T_{n-1} \cup PA \cup \{ \gamma \}_{i \in E \} \cup \{ \gamma \}_{i \notin E \} \).

As our construction is recursive in \( d \) it is clear that \( T = d \).

It also follows that \( T \) is a complete extension of \( PA \), \( T \cap \Sigma_n \in S \) for each \( n \in \omega \), and \( E_m \leq T \cap \Sigma_{4m+3} \cdot \) Since each \( T \cap \Sigma_n \in S \), \( \text{Rep}(T) \subseteq S \). But since each \( E_m \in S \) is recursive in some \( T \cap \Sigma_n \) and \( \text{Rep}(T) \) is a Scott set, \( S \subseteq \text{Rep}(T) \). Thus \( \text{Rep}(T) = S \).

In Section 7 we will sharpen this result to give a necessary and sufficient condition.

Earlier we wondered what effect our choice of \( \text{Re}(M) \) has in determining \( \text{Th}(M) \). The next easy corollary shows it is very small.

**Corollary 1.28**: Let \( S \) be a Scott set. There are distinct \( \{ M_\alpha : \alpha < 2 \} \) such that \( M_\alpha \) is minimal and \( \text{Re}(M_\alpha) = S \). In particular, if \( \alpha \neq \beta \) \( M_\alpha \neq M_\beta \).

In fact, even after we have fixed \( \text{Re}(M) \) and \( \text{Th}(M) \) we are a long way from characterizing \( M \).
Proposition 1.29: Let $S$ be a Scott set and let $T \neq \text{Th}(\mathbb{N})$ be complete such that $\text{Rep}(T) = S$. There are $\{M_\alpha : \alpha < 2^\omega_0\}$ such that for each $\alpha, \beta \in \omega_0$ such that for each $\alpha, \beta \in \omega_0$ such that for each $\alpha, \beta \in \omega_0$, $M_\alpha \models T$, $\text{Re}(M_\alpha) = S$ and $|M_\alpha| = 2^\omega$ but if $\alpha \neq \beta$ then $M_\alpha \not\cong M_\beta$.

Proposition 1.29 follows from a deep theorem of Gaifman via a simple observation.

Lemma 1.30: If $M, N \models \text{PA}$ and $N$ is an end extension of $M$ then $\text{Re}(M) = \text{Re}(N)$.

Proof: Let $a \in N$ and let $b \in M$. Consider the type $\Gamma(v) = \{p_n | v \leftrightarrow p_n | a : n \in \omega\} \cup \{v < b\}$. This is consistent and hence realized by a $c \in M$. Then $r(a) = r(c)$.

Theorem 1.31: (Gaifman [G]). If $M \models \text{PA}$ is minimal, there are $2^{2^\omega}$ nonisomorphic countable elementary end extensions of $M$.

In Section 5 we will restrict our considerations to recursively saturated models. Here $\text{Re}(M)$ and $\text{Th}(M)$ determine $M$ up to isomorphism.
§2. DEGREES CODING COMPLETE EXTENSIONS OF PA.

In this section we will survey some of the known results on degrees coding models of arithmetic. Tennenbaum's theorem (Corollary 1.9) shows that if $M$ is a nonstandard model of $PA$, then $\deg(M) \notin \mathbb{Q}$. More precisely, there is $X \preceq^T_{Diag} M$ such that $X$ separates a pair of effectively inseparable r.e. sets. (Recall r.e. sets $A$ and $B$ are effectively inseparable if there is a partial recursive function $\psi(x,y)$ so that if $A \subseteq W_x$, $B \subseteq W_y$ and $W_x \cap W_y = \emptyset$, then $\psi(x,y) \in \omega (W_x \cup W_y)$. The sets $A_\emptyset$ and $B_\emptyset$ used in 1.6 are effectively inseparable as are the sets $\{\varphi \vdash PA \varphi\}$ and $\{\varphi \vdash PA \neg \varphi\}$. See Rogers [R] or Smullyan [Smu] for more information.) We have also shown that $\text{Diag}(M)$ enumerates a Scott set. The following theorem gives a complete classification of degrees coding nonstandard models of $PA$. It appears (with the exception of iv) in Simpson [Si] where it is attributed to a combination of Scott and Tennenbaum [Sc-T], Solovay and Jockusch and Soare [J-S].

**Theorem 2.1:** The following are equivalent.

i) There is a nonstandard $M \models PA$ such that $\text{Diag}(M) \equiv^d T$.

ii) There is a complete extension $T$ of $PA$ such that $T \equiv^d T$.

iii) There is $X \equiv^d_T$ which separates a pair of effectively inseparable r.e. sets.

iv) There is a d effective enumeration of some Scott set $S$.

Before embarking on the proof of 2.1 we prove a useful padding lemma.

**Lemma 2.2:** If $M \models PA$ and $e_T \geq \text{Diag}(M)$, then there is $N \equiv M$ with $\text{Diag}(N) \equiv^e_T$. 

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Proof: Let $M = (\omega, \oplus, \otimes)$. We can find an $e$-recursive bijection $f : \omega \rightarrow \omega$ such that for any $n \in \omega$ $n \in e$ iff $M \models "f(m) is even."$

Define $\oplus'$ and $\otimes'$ so that

$$m_0 \oplus' m_1 = m_2 \text{ iff } f(m_0) \otimes f(m_1) = f(m_2)$$

and

$$m_0 \otimes' m_1 = m_2 \text{ if } f(m_0) \otimes f(m_1) = f(m_2).$$

Let $N = (\omega, \oplus', \otimes')$. Clearly, $f : N \rightarrow M$ is an isomorphism and $\text{Diag}(N) \equiv_T e$. //

We can now prove the equivalence of i), ii) and iv).

i) = iv) This is Solovay's Lemma 1.14.

iv) = ii) This follows from Theorem 1.27.

ii) = i) Given any $T$ by an effective Henkin argument we can find $M \models T$ such that the full diagram ($\text{Diag}^*(M)$) of $M$ is recursive in $T$. By Lemma 2.2 we can pad $M$ to $N \models T$ with $\text{Diag}(N) \equiv_T T$.

A corollary to this equivalence is the following unpublished result of Solovay.

**Corollary 2.3:** If $T$ is a complete extension of PA and $e_T \geq T$, there is another complete extension $T'$ where $T' \equiv_T e$.

**Proof:** This follows from 2.2 and the i) = ii) portion of 2.1.

(For the sake of completeness we would like to sketch Solovay's original proof) Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ list all sentences in the language of PA. At any stage $s$ we will have formed a sentence $\psi_s$ a conjunction of $\varphi_i$ or $\neg \varphi_i$ $i \leq s$. Assume at stage $s = 2m$ $\psi_{s-1}$ is consistent with PA. Ask $T$ if $\psi_{s-1} \land \varphi_m$ is consistent with PA. If $T \models \text{Con}(\psi_{s-1} \land \varphi_m)$, then $\psi_s$ is consistent with PA as $\Pi^0_1$ sentences are preserved under
initial segments. If $T \vdash \text{Con}(\phi_{s-1} \land \psi_s \land \neg \phi_m)$, ask if $T \vdash \text{Con}(\phi_{s-1} \land \neg \phi_m)$. If it does, let $\psi_s = \phi_{s-1} \land \phi_m$. Suppose $T \vdash \neg \text{Con}(\phi_{s-1} \land \phi_m) \land \text{Con}(\phi_{s-1} \land \neg \phi_m)$. In $T$ we can prove there are $x$ and $y$ such that $x$ and $y$ are the least Godel codes of a proof of $0 = 1$ from $\phi_{s-1} \land \phi_m$ and $\phi_{s-1} \land \neg \phi_m$, respectively. As $\psi_s$ is consistent with PA at most one of $x$ and $y$ may be standard. Thus if we add $\phi_m$ to $\psi_{s-1}$ to maximize the code of the shortest proof of a contradiction, we maintain consistency. This step can be done recursively in $T$.

At stage $s = 2m+1$ we use the independent sentences from 1.20 as we did in 1.21 to code $e$ into $T$. (The entire construction can be done recursively in $e$.)

Another corollary to the equivalences we have proved so far is the following corollary which generalizes the Scott-Tennenbaum result that no complete extension of PA has minimal degree. (Jockusch and Soare [J-S] have shown that any countable partial ordering can be embedded in the Turing degrees below a complete extension of PA.)

**Corollary 2.4:** If $T$ is a complete extension of PA, then there is another complete extension $T'$ such that $T' <_T T$.

**Proof:** By ii) = iv) there is a Scott set $S$ with a $T$ effective enumeration. But by 1.3 $S$ contains $T'$ a complete extension of PA. By 1.6 $T' <_T T$.

Continuing the proof of 2.1, we notice that our proof of 1.9 shows i) = iii). We will finish the proof by showing iii) = iv). We use the following theorem of Smullyan.
Theorem 2.5: (Smullyan [Smu]). Suppose \((A,B)\) and \((C,D)\) are two pairs of effectively inseparable r.e. sets. Then there is a recursive bijection \(f: \omega \to \omega\) such that \(f(A) = C\) and \(f(B) = D\). Moreover, given indices for \((A,B,C,D)\) and the functions witnessing effective inseparability, we can effectively find an index for \(f\).

Proof of iii) = i): Let \(\varphi_1, \varphi_2, \varphi_3, \ldots\) list all sentences in the language of PA. Let \((A,B)\) be a pair of effectively separable r.e. sets. Let \(\psi_0(x,y)\) witness this. Let \(X =_{\text{d}}\) be such that \(A \subseteq X\) and \(B \cap X = \emptyset\). Let \(P_0 = \{\varphi: \text{PA} \vdash \varphi\}\) and \(R_0 = \{\varphi: \text{PA} \vdash \varphi\}\). As noted above \(P_0\) and \(R_0\) are effectively inseparable. Let \(\psi_1(x,y)\) witness this.

We build a complete theory recursively in \(X\). At each step \(s\) we build \(\theta_s\) a finite conjunction of \(\varphi_i\), \(i \leq s\), and r.e. sets \(P_s = \{\varphi: \text{PA} \vdash \theta_s \rightarrow \varphi\}\), \(R_s = \{\varphi: \text{PA} \vdash \theta_s \rightarrow \neg \varphi\}\). As long as \(\theta_s\) is consistent with PA, \(P_s\) and \(R_s\) are effectively inseparable r.e. sets. Note that \(\psi_1(x,y)\) is still a witness for the inseparability of \(P_s\) and \(R_s\). Thus by 1.4 we can effectively find a recursive bijection \(f: \omega \to \omega\) such that \(f(A) = P_s\) and \(f(B) = R_s\). Look at \(\varphi_{s+1}^*\).

Recursively in \(X\) we can decide if \(\varphi_{s+1} \in f(X)\). If it is, let \(\theta_{s+1} = \theta_s \land \varphi_{s+1}\). As \(f(X)\) separates \(P_s\) and \(R_s\), \(\text{PA} \not\vdash \theta_s \rightarrow \neg \varphi_{s+1}\), so \(\theta_{s+1}\) is consistent with PA. If \(\varphi_{s+1} \notin f(X)\), let \(\theta_{s+1} = \theta_s \land \neg \varphi_{s+1}\). As \(\theta_{s+1} \notin f(X)\), \(\text{PA} \not\vdash \theta_s \rightarrow \varphi_{s+1}\). Thus \(\theta_{s+1}\) is consistent.

Let \(T = \{\theta: s \in \omega\}\). \(T\) is a complete extension of PA and \(T \leq_{\text{d}} \). By Corollary 2.3 we can find another complete \(T^*\) such that \(T^*_s \in T\).

This concludes the proof of Theorem 2.1.

Theorem 2.1 can be extended in several ways.
Definition 2.6: If $M = (\omega, \mathbb{N}, 0) \models PA$, let $\text{Diag}^*(M)$ denote the full diagram of $M$. Let $\text{deg}^*(M)$ be the Turing degree of $\text{Diag}^*(M)$.

It is easily seen that the $ii) = i)$ proof of 2.1 gives $M \models PA$ with $\text{Diag}^*(M) \equiv^d_T$. Further, 2.2 shows that if there is $M \models PA$ with $\text{Diag}^*(M) \equiv^d_T$, there is $N \cong M$ with $\text{Diag}(M) \equiv^d_T$. Thus we could add to 2.1 the equivalence, $v)$. There is a nonstandard $M \models PA$ with $\text{Diag}^*(M) \equiv^d_T$.

We will show in Section 7 that we may also add the equivalence $vi)$. There is a Scott set $S$ with a d-enumeration.
§3. DEGREES OF MODELS OF $\text{Th}(\mathbb{N})$

In 2.1 we characterized the degrees coding some nonstandard model of $\text{Th}(\mathbb{N})$. Suppose we fix $T$ a completion of PA. What degrees code nonstandard models of $T$? One negative restriction is apparent.

Proposition 3.1: If $M \models \text{PA}$, then $\text{deg}(M)^{(\omega)}_{T} \geq \text{Th}(M)$.

Proof: This is clear since $\text{Th}_{\Sigma_{n+1}}(M) \leq_{T} (\text{Th}_{\Sigma_{n}}(M))^{'},$ uniformly in $n$. //

This bound seems somewhat loose as we know that if $M$ is nonstandard each $\text{Th}_{\Sigma_{n}}(M) \leq_{T} \text{deg}(M)$. But Harrington [H] showed that it is the best possible.

Theorem: (Harrington [H]). There is a nonstandard $M \models \text{PA}$ such that $\text{Diag}(M) \leq 0'$, while $\text{Th}(M) \equiv_{T} 0^{(\omega)}$.

On the positive side, if $T \neq \text{Th}(\mathbb{N})$ and $e \geq_{T} T$, then the usual Henkin arguments and Lemma 2.2 guarantee the existence of a nonstandard $M \models T$ with $\text{Diag}(M) \leq_{T} e$. This is still true for $\text{Th}(\mathbb{N})$ though we must excercise some care to insure that the model produced is nonstandard. Knight showed that this condition is unnecessary for $\text{Th}(\mathbb{N})$.

Theorem 3.3: (Knight [K2]). There is a nonstandard $M \models \text{Th}(\mathbb{N})$ with $\text{Diag}(M) \leq_{T} 0^{(\omega)}$.

Proof: Let $C = \{c_{k} : k \in \omega\}$ be a set of Henkin constants. We will build $M$ from the constants $C$. Our construction of $M$ will be recursive in $0^{(\omega)}$. Let $T_{0} = \text{Th}(\mathbb{N}) \cup \{c_{0} > n : n \in \omega\}$. Note that, by
overspill, a sentence $\varphi(c_0, \ldots, c_n)$ is consistent with $T_0$ iff  
$\forall y \exists x_0 > y \exists x_1 \ldots \exists x_n \varphi(x_0, \ldots, x_n) \in \text{Th}(N)$. Thus we can test consistency with $T_0$ recursively in $0^\omega$. We will build consistent theories $T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$ such that  
(i) $T_{n+1}$ is a finite extension of $T_n$.  
(ii) $\mathcal{U}T_n$ is complete and Henkinized.  
(iii) For each $e \in \omega$ there is a finite set of basic sentences $X$ and an $m \in \omega$ such that $X \subseteq T_m$ and for some $k \in \omega$ either:  
   a) $\{e\}^X(k) = 1$ and $k \notin 0^\omega$;  
   b) $\{e\}^X(k) = 0$ and $k \in 0^\omega$; or  
   c) for all $X' \supseteq X$ included in any $T_n$ $\{e\}^{X'}(k)$ does not converge.

Requirement (ii) is handled in the usual manner. Suppose at stage $n$ we want to handle requirement (iii). Suppose this cannot be done. Then there is a finite set of basic sentences $X'$ such that for any $k \in \omega$, $k \in 0^\omega$ iff there is a finite set of sentences $X \supseteq X'$ such that $T_n \cup X$ is consistent and $\{e\}^X(k) = 1$. We see from this that there is a formula $\varphi(v)$ in the language of arithmetic such that $k \in 0^\omega$ iff $N \models \varphi(k)$. But then $0^\omega$ is arithmetic, a contradiction. Thus we can always find an $X$ and $k$ to satisfy (iii).

Let $T_{n+1} = T_n \cup X$.

Let $M$ be the Henkin model of $\mathcal{U}T_n$. Condition (iii) insures $0^\omega \models T > \text{Diag}(M)$. Hence $0^\omega > T > \text{Diag}(M)$.

Fefferman's result (1.18) shows that if $M \models \text{Th}(N)$ is nonstandard, then $\text{deg}(M)$ is greater than every arithmetic degree. Jockusch asked what other restrictions could be placed on $\text{deg}(M)$ and Knight
conjectured there were none.

**Knight's Conjecture:** If for all $n \in \omega$, $d_T \geq 0^{(n)}$, then there is $M \models \text{Th}(\mathbb{N})$ such that $\text{Diag}(M) \equiv_T d$.

In [K-L-S], Knight–Lachlan–Soare shows that Knight's conjecture is false, by producing a degree $d'$ above all the arithmetic degrees such that if $E \subseteq \omega^2$ and $E \leq d$, the $\{E_n : n \in \omega\}$ does not contain all the arithmetic sets. (In the language of recursion theory $d'$ is not a subuniform upperbound for the arithmetic degrees.) By 1.17 this shows there is no nonstandard $M \models \text{Th}(\mathbb{N})$ with $\text{deg}(M) = d$. Recently Lerman [L] has shown that there is a degree $d''$ above all the arithmet- metic degrees such that $d'' \geq Q^{(\omega)}$. (The Knight–Lachlan–Soare example has $d'' = 0^{(\omega)}$.) This refutes a conjecture of the author from [M].

Before Knight's conjecture was refuted, there were several positive results. Before stating them let us recall some facts about the degrees below $Q^{(\omega)}$. For proofs see [E].

**Theorem 3.4:** i) (Enderton–Putnam). If for all $n \in \omega$ $d \geq 0^{(n)}$, then $d'' \geq Q^{(\omega)}$.

ii) (Sacks). There is a $d''$ such that for all $n \in \omega$ $d \geq 0^{(n)}$ and $d'' = Q^{(\omega)}$.

**Theorem 3.5:** (Marker [M]). If for all $n \in \omega$ $d_T \geq 0^{(n)}$, then there is a nonstandard $M \models \text{Th}(\mathbb{N})$ such that $\text{Diag}(M) \equiv_T d'$.

Section 4 is devoted to the proof of 3.5. Theorem 3.5 and 3.4 ii) give the following immediate corollary.

**Corollary 3.6:** There is a nonstandard $M \models \text{Th}(\mathbb{N})$ such that $(\text{Diag}(M))'$ $\equiv_T 0^{(\omega)}$. 
Corollary 3.6 can be thought of as an analog of the Jockusch and Soare result that there are nonstandard $M \models PA$ with $(\text{Diag}(M))' \equiv_T 0'$. Theorem 3.5 and Corollary 3.6 were strengthened by Knight-Lachlan-Soare.

**Theorem 3.7**: (Knight-Lachlan-Soare [K-L-S]). If for any $n \in \omega$

$\exists d \succ 0^{(n)}$, then there is a nonstandard $M \models \text{Th}(\mathbb{N})$ with $(\text{Diag}(M))' \equiv_T d'$.

**Corollary 3.8**: There is a nonstandard $M \models \text{Th}(\mathbb{N})$ such that $\text{Diag}(M)'' \equiv_T 0^{(\omega)}$.

These results still shed little light on Jockusch's problem.

Solovay provided the final solution.

**Theorem 3.9**: (Solovay [So]). Let $S$ be a Scott set containing all the arithmetic sets. There is $M \models \text{Th}(\mathbb{N})$ with $\text{Re}(M) = S$ and $\text{Diag}(M) \equiv_T d$ iff there is a $d$ effective enumeration of $S$.

This will be simplified in Section 7.
§4. THE PROOF OF THEOREM 3.5

This section is devoted to the proof of Theorem 3.5. This proof uses a construction first used by Harrington in his proof of 3.2. We will actually show that if for every $n \in \omega$ $d \uparrow n \geq o(n)$, then there is a nonstandard $M \models \text{Th}(\mathbb{N})$ such that $\text{Diag}(M) \leq d'$. Theorem 3.5 follows from this using Lemma 2.2.

We will describe a three worker construction which produces a complete, consistent, Henkinized theory $T$ such that $M$ is the canonical model of $T$. Worker 1 will use oracle $d'$ and produce the $\Sigma_1$-diagram of $M$. Worker 2 will use oracle $d''$ and build the $\Sigma_2$-diagram of $M$. Worker 3 uses oracle $d'''$ and constructs the full diagram of $M$. As every arithmetic set is recursive in $d$, $\phi(\omega) \leq d''$ by 3.4 (i). Thus workers 2 and 3 each have access to $\text{Th}(\mathbb{N})$.

Let $L$ be the language of Peano Arithmetic. Let $C = \{c_i : i \in \omega\}$ be a set of Henkin constants. For $i = 1, 2$, let $\{\varphi_j^i : j \in \omega\}$ be a recursive listing of all $\Sigma_i$-sentences in $L(C)$. Let $\{\varphi_j^3 : j \in \omega\}$ be a recursive listing of all $L(C)$ sentences. We denote the set of Boolean combinations of $\Sigma_n$-formulas as $b\Sigma_n$. $B_n(S)$ will denote all $b\Sigma_n$ consequences of $S$ allowing no constants from $C$ not already mentioned in $S$. For $i = 1, 2$, let $\{\Gamma_j^i(v) : j \in \omega\}$ list all sets $\Gamma$ of $b\Sigma_i$ formulas allowing one free variable $v$ and finitely many constants from $C$ such that $\Gamma$ is r.e. in $d$.

We will assume, via the recursion theorem, that each worker knows the others' strategies. That is, we actually describe a recursive function $g$ such that if $x$ is an index for the strategy used by worker $n+1$, $g(x)$ will be an index for the strategy used by worker $n$. 

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By the recursion theorem there is a natural number \( e \) s.t. for any oracle \( W^A_e = W^A_{E(e)} \). We use the strategy given by index \( e \).

By any stage \( s \) worker \( j \) will have committed himself to a finite conjunction \( T^j_s \), chosen from \( \{\varphi^j_i, \forall^j_i: i \leq s\} \). \( T^j \) will denote \( \{T^j_s: s \in \omega\} \). We will arrange things so that \( T^1 \subseteq T^2 \subseteq T^3 \), \( \text{Th}(N) \subseteq T^3 \) and \( T^3 \) is complete, consistent and Henkinized.

As \( d^{(n+1)} \) is r.e. in \( d^{(n)} \), worker \( n \) may enumerate worker \( n+1 \)'s oracle and approximate the actions of worker \( n+1 \). For \( n = 1,2 \) let \( k^{n+1}_{i,j} \) denote worker \( n \)'s approximation to \( T^{n+1}_{i} \) based on the \( j \)th approximation to \( d^{(n+1)} \). To insure our computation of \( k^{n+1}_{i,j} \) converges we continue enumerating \( d^{(n+1)} \) and dovetail computations using better approximations to \( d^{(n+1)} \). As worker \( n+1 \)'s computations of \( T^{n+1}_{i} \) converge, we will eventually find a convergent computation of \( k^{n+1}_{i,j} \). In time we will have enumerated enough of \( d^{(n+1)} \) to correctly answer all of worker \( n+1 \)'s stage \( i \) oracle queries. Thus, for each \( i \) there is an \( s \) such that, for any \( s' \geq s \)

\[
k^{n+1}_{i,s'} = T^{n+1}_{i}.
\]

For any \( s \) and \( i < s \), worker 2 will form a set \( U^2_{i}(s) \) which is r.e. in \( d \). (To be perfectly correct, worker 2 will find \( u^2_i(s) \in \omega \) which is an index for \( U^2_{i}(s) \) in \( d \). We will not distinguish between \( U^2_{i}(s) \) and \( u^2_i(s) \) unless significant confusion arises.) \( U^2_{i}(s) \) will contain \( E_2(\text{Th}(N) \cup K^3_{i,s}) \). Note that for some \( m \in \omega \) \( K^3_{i,s} \) is a set of \( \Sigma_m \)-sentences. Thus \( E_2(\text{Th}(N) \cup K^3_{i,s}) = \{\varphi \in \Sigma_2^m: \text{all constants in } \varphi \text{ are contained in } K^3_{i,s} \text{ and } \text{Th}_{\Sigma_m}(N) \models \forall^c(K^3_{i,s} \rightarrow \varphi)\} \). As \( \text{Th}_{\Sigma_m}(N) \leq_T d \), \( E_2(\text{Th}(N) \cup K^3_{i,s}) \) is r.e. in \( d \).

If \( V^2_1(v) \) ends up consistent with the full diagram of \( M \), there will be a large \( s' \) and \( c \in C \) such that for any \( s \geq s' \) \( U^2_{i}(s) \geq V^2_1(c) \).
and worker 2 has oracle $d''$, worker 2 could maintain consistency of $T^1 \cup \text{Th}(\mathbb{N}) \cup T^2$. (Note that as $T^1 \subseteq \Sigma_1$ and $T^1 \cup \text{Th}_\Sigma_2(\mathbb{N})$ is consistent, $T^1 \cup \text{Th}(\mathbb{N})$ is consistent.) Similarly, worker 3 could insure $\text{Th}(\mathbb{N}) \cup T^2 \cup T^3$ is consistent, while completing $T^3$.

The difficulty arises in insuring that $T^3$ is Henkinized. For example, suppose worker 2 writes down $\exists x \forall y \varphi(x,y,c)$ where $\varphi$ is quantifier free. In order to provide a witness for $\forall y \varphi(x,y,c)$ we must insure worker 1 has set aside a constant $c_0$ such that $\exists y(c_0,y,c)$ is not in $T^1$. Similarly, if worker 3 writes down $\exists x \forall y \exists z \varphi(x,y,z,c)$, worker 2 must set aside a $c_0$ such that $\exists y \forall z \varphi(c_0,y,z,c)$ is not in $T^2$ and for any $c_1$ worker 1 must insure $\exists z \varphi(c_0,c_1,z,c)$ is in $T^1$.

This difficulty is overcome by our approximation procedures. If worker 2 writes down $\exists x \forall y \varphi(x,y,c)$, there will be a later stage where worker 1 realizes that worker 2 wants a witness to $\forall y \varphi(x,y,c)$. At this point worker 1 will find a $c_0$ such that none of the workers could have considered $c_0$ by this point and set $c_0$ aside as a witness (i.e., worker 1 will not write down $\exists y \varphi(c_0,y,c)$). As worker 2 will realize that worker 1 has done this, worker 2 may at some point write down $\forall y \varphi(c_0,y,c)$. From this point on everyone is committed to this choice.

Providing a witness for a $\Sigma_n^0$-sentence $\exists x \psi(x)$ where $n > 2$, is a bit more complex. First, worker 2 must provide a witness for the $\Sigma_2$-consequences of $\psi(x)$. Secondly, this witness must also have been provided by worker 1 as a witness to the $\Sigma_1$-consequences of $\psi(x)$. To insure this occurs, workers 1 and 2 attempt to partially saturate the model. Namely, if they believe it is consistent with the actions of higher level workers, they will set aside a witness for $t^3_1(v)$. 
We will arrange things so that for each \( i \), there is a \( t \) such that if \( s \geq t \) \( U_i^2(s) = U_i^2(t) \) (in fact, \( u_i^2(s) = u_i^2(t) \)). \( U_i^2 \) will denote \( U_i^2(s) \) for such an \( s \).

Worker 1 will calculate \( K_{i,s}^2 \) approximating \( T_i^2 \). He will also calculate \( \delta_j(s,t) \), an approximation to \( u_j^2(s) \) based on the \( t^{th} \) approximation to \( d'' \). \( \Delta_j(s,t) \) will be the set r.e. in \( d \) with index \( \delta_j(s,t) \). At some stage \( t \), worker 1 will have enumerated a sufficient portion of \( d'' \) to guarantee that for any \( t' \geq t \) \( \delta_j(s,t') = u_j^2(s) \). As above, whenever possible we suppress \( \delta_j(s,t) \) and concentrate on \( \Delta_j(s,t) \).

For \( i < s \), worker 1 will form a set \( U_{i}^1(s) \) r.e. in \( d \) (again, we really find an index \( u_{i}^1(s) \) for \( U_{i}^1(s) \)). \( U_{i}^1(s) \) will contain \( B_i(N) \cup K_{i,s}^2 \cup \Delta_1(i,s) \cup U_{i-1}^1(i,s) \). Note that, as above, this is r.e. in \( d \) and in fact, given indices for \( \Lambda_1(i,s) \ldots \Lambda_{i-1}(i,s) \) and \( K_{i,s}^2 \), we can compute an index for \( B_i(N) \cup K_{i,s}^2 \cup \Delta_1(i,s) \cup U_{i-1}^1(i,s) \) effectively in \( d' \). If \( \Gamma_i^1(v) \) is eventually consistent with the full diagram of \( M \), there will be a constant \( c \in C \) and a stage \( s' \) such that if \( s' \geq s \), then \( U_{i}^1(s') \supseteq \Gamma_i^1(c) \). Again we will arrange things so that \( \lim_{s} u_{i}^1(s) \) exists. \( U_{i}^1 \) will denote \( \lim_{s} u_{i}^1(s) \).

The basic ideas:

Before detailing the construction, we should outline some of the ideas behind it.

There would be no difficulties involved if we only had to maintain the consistency of \( T_1^1 \), \( T_2^2 \) and \( T_3^3 \) while completing \( T_3^3 \). Worker 1 would make sure \( Th_{\omega_4}(N) \cup T_s^1 \) is consistent for each \( s \). As \( T_1^1 \leq T_4^4 \),
Witnessing $\Gamma^j_{11}(v)$ is given priority over witnessing $\Gamma^j_{1k}(v)$ for $i < k$, to make sure all guesses settle down. If it is consistent for worker 2 to realize $\Gamma^2_{11}(v)$, then it must be consistent for worker 1 to realize $B_1(\Gamma^2_{11}(v))$ so worker 1 will have done so. Thus, worker 2 will be able to choose a witness. Similarly, if it is consistent for worker 3 to write down $\exists x \psi(x)$, then it must have been consistent for worker 2 to realize $B_2(\psi(v))$. Hence, worker 2 must have done so.

One final point should be made. As worker 1 does not have access to all of $\text{Th}(\mathbb{N})$, he can not uniformly compute the consequences of formulas of arbitrary complexity. For this reason we have the intermediate worker 2, which, while it could maintain consistency with $\text{Th}(\mathbb{N})$, restricts itself to producing the $\Sigma^2_2$-theory to keep worker 1 happy. Worker 3 may produce sentences of arbitrary complexity, since worker 2 has the resources to compute their consequences.

The construction:

Worker 1: Stage s.

Worker 1 first enumerates a bit more of $d''$. As worker 1 knows, worker 2 will be maintaining consistency, worker 1 will enumerate enough of $d''$ to insure that the computations of $K^2_{i,s}$ and $\Delta_1(i,s) \Delta_{i-1}(i,s)$ converge for $i \leq s$, and that $K^2_{i,s} \cup \Delta_1(i,s) \cup \ldots \cup \Delta_{i-1}(i,s)$ is consistent for $i \leq s$. Here, by consistent, we mean consistent with $\text{Th}(\mathbb{N})$. At first it may seem that this task is beyond worker 1's abilities as he only has $d'$ as an oracle. But if $X \subseteq b\Sigma^2_2$ then $X \cup \text{Th}(\mathbb{N})$ is consistent iff $X \cup \text{Th}_{\Sigma^2_4}(\mathbb{N})$ is consistent. As $\text{Th}_{\Sigma^2_4}(\mathbb{N}) \equiv_T d'$, this can be checked effectively in $d'$. 
If $\varphi_s^1$ is quantifier free, worker 1 sets $T_s^1 = T_{s-1}^1 \land \varphi_s^1$ or $T_s^1 = T_{s-1}^1 \land \neg \varphi_s^1$ to maintain consistency of
\[(*) \quad \text{Th}_\frac{2}{4}(\text{Q}) \cup K_{i,s}^2 \cup \Delta_1(i,s) \cup \cdots \cup \Delta_{s-1}(i,s) \cup U_1(s-1) \cup \cdots \cup U_{s-1}(s-1) \cup T_s^1,
\]
for all $i \leq j \leq s$ for the largest possible $j$. If $\varphi_s^1$ is not quantifier free, worker 1 sets $T_s^1 = T_{s-1}^1 \land \varphi_s^1$ or $T_s^1 = T_{s-1}^1$ to maintain consistency of $(*)$ for all $i \leq j \leq s$.

We next define $U_i^1(s)$, $i < s$. Let $X_i^S$ denote $K_{i,s}^2 \cup \Delta_1(i,s) \cup \cdots \cup \Delta_{s-1}(i,s) \cup \text{Th}_\frac{2}{4}(\text{Q}) \cup U_i(s) \cup U_{i-1}(s-1) \cup T_i^1$.

**Case 1:** $i < s-1$ and $T_s^1 \cup X_i^S \cup \Gamma_i^1(v) \cup U_i^1(s-1)$ is consistent.

a) If $U_i^1(s-1)$ contains a realization of $\Gamma_i^1(v)$, then the set $U_i^1(s) = B_1(X_i^S \cup U_i^1(s-1))$.

b) If $U_i^1(s-1)$ does not realize $\Gamma_i^1(v)$, find a constant $c$ which no worker could not have used by this stage. Let $U_i^1(s) = B_1(X_i^S \cup U_i^1(s-1) \cup \Gamma_i^1(c))$.

**Case 2:** $i < s-1$, $T_s^1 \cup X_i^S \cup \Gamma_i^1(v) \cup U_i^1(s-1)$ is inconsistent. Let $U_i^1(s) = B_1(X_i^S \cup \Gamma_i^1(c))$, where $c$ is a constant which no worker could have used by this stage.

**Case 3:** $i < s-1$ and $T_s^1 \cup X_i^S \cup \Gamma_i^1(v)$ is inconsistent.

If $T_s^1 \cup X_i^S \cup U_i^1(s-1)$ is consistent, let $U_i^1(s) = B_1(X_i^S \cup U_i^1(s-1))$. Otherwise let $U_i^1(s) = B_1(X_i^S)$.

**Case 4:** $i = s-1$. (In this case $U_{s-1}^1(s-1)$ is not defined). a) If $T_s^1 \cup X_i^S \cup \Gamma_i^1(v)$ is consistent, let $c$ be a new constant and let
\[ U^1_i(s-1) = B_1(x^s_i \cup \Gamma^1_i(c)). \]

b) If \( T^1_s \cup x^s_1 \cup \Gamma^1_i(v) \) is inconsistent, let \( U^1_i(s) = B_1(x^s_1) \).

This concludes worker 1's construction.

There are several things to notice.

- Our program is to maintain consistency, while, if at all possible, realizing the types \( \Gamma^1_i(v) \). We give priority to realizing types of lower index. We also give priority to our earlier witnesses.

- By induction it is easily seen that each \( U^1_i(s) \) is r.e. in \( d \).

- Suppose at stage \( s_0 \), \( U^1_{1}, \ldots, U^1_{i-1} \) have settled down and for \( s \geq s_0 \), \( x^2_1, s = T^2_j \) and \( U^2_k(j) = \Delta_k(j,s) \) for all \( k < j \leq i \). Then from this stage onward \( x^s_1 \) is fixed. Since for \( s > s_0 \) \( T^1_s \) will be chosen to maintain consistency with \( U^1_i(s_0) \), once we make the decision to omit \( \Gamma^1_i(v) \) or realize it with \( c \), we will never be forced to back down from this choice. Thus for \( s \geq s_0 + 1 \), \( U^1_i(s) = U^1_i(s_0 + 1) \). Thus \( \lim_{s} U^1_i(s) \) exists.

Worker 2: At stage \( s \) worker 2 will be in either the "active" mode or the "waiting" mode.

Worker 2 knows that for each \( i \) there is an \( s \) such that if \( s' \geq s \), \( U^1_i(s') = U^1_i(s) \). The condition \( \forall s' \geq s \ U^1_i(s') = U^1_i(s) \) is recursive in \( d'' \) so worker 2 may calculate indices for \( U^1_{1}, \ldots, U^1_{s-1} \).

Worker 2 enumerates more of \( d''' \) and calculates \( k^3_{i,s}, i \leq s \). As worker 3 is maintaining consistency, worker 2 may enumerate enough of \( d''' \) to insure the consistency of \( T^1 \cup U^1_1 \cup \ldots \cup U^1_{s-1} \cup T^2_{2-1} \cup K^3_{i,s} \cup \text{Th}(\mathbb{N}) \) for \( i \leq s \).

Case I: Worker 2 is in the active mode.
Worker 2 considers the next $\varphi^2_i$ which has not been attended to and sets $T^2_s = T^2_{s-1} \land \varphi^2_i$ or $T^2_s = T^2_{s-1}$ to keep $Th(N) \cup T^1_s \cup T^2_s \cup U^1_{s-1} \cup \cdots \cup U^1_i \cup K^3_{i,s} \cup U^2_{i-1}(s-1) \cup \cdots \cup U^2_i(s-1)$, $i \leq j$ consistent for as large a $j \leq s$ as is possible.

We must define $U^2_i(s)$ for $i < s$. Let $\chi^s_i$ denote $T^2_i \cup K^3_{i,s} \cup U^2_i(s) \cup \cdots \cup U^2_{i-1}(s)$. Let $\psi_s$ denote $T^1 \cup U^1_s \cup U^1_{s-1} \cup T^2_s$.

**Case 1**: $i < s-1$ and $\psi_s \cup \chi^s_i \cup \Gamma^2_\ell(v) \cup U^2_i(s-1)$ is consistent.

a) If $U^2_i(s-1)$ contains a witness for $\Gamma^2_\ell(v)$, then set $U^2_i = B_2(\chi^s_i \cup U^2_i(s-1))$.

b) If $U^2_i(s-1)$ contains no realization of $\Gamma^2_\ell(v)$, we calculate $B_1(T^2_{s-1} \cup \Gamma^2_\ell(v) \cup U^2_i(s) \cup \cdots \cup U^2_{i-1}(s))$. This is r.e. in $d$ and thus is $\Gamma^1_\ell(\omega)$ for some $\ell \in \omega$. In $d'$ we may calculate $\ell$. Worker 2 sets $U^2_j(s) = B_2(\chi^s_j)$ for $i \leq j < s$ and goes into the waiting mode to find a witness for $\Gamma^1_\ell$.

**Case 2**: $i < s-1$, $\psi_s \cup \chi^s_i \cup \Gamma^2_\ell(v)$ is consistent while $\psi_s \cup \chi^s_i \cup \Gamma^2_\ell(v) \cup U^2_i(s-1)$ is inconsistent. In this case we act as in lb.

**Case 3**: $\psi_s \cup \chi^s_i \cup \Gamma^2_\ell(v)$ is inconsistent.

a) if $i < s-1$ and $\psi_s \cup \chi^s_i \cup \Gamma^2_{i}(s-1)$ is consistent, let $U^2_i(s) = B_2(\chi^s_i \cup U^2_i(s-1))$.

b) otherwise let $U^2_i(s) = B_2(\chi^s_i)$.

**Case 4**: If $\psi_s \cup \chi^s_i \cup \Gamma^2_\ell(v)$ is consistent and $i = s-1$, proceed as in case lb.

**Case II**: Player 2 is waiting for a witness to $\Gamma^1_\ell(v)$ which was demanded at stage $s_0$. 
If \( s \leq \ell \), then set \( T^2_s = T^2_{s_0} \), \( U^2_j(s) = U^2_j(s_0) \) for \( j < s_0 \) and \( U^2_i(s) = \emptyset \) for \( s_0 \leq i < s \). If \( s > \ell \), we look at \( U^1_{\ell} \) to see if it contains a witness \( c \) for \( \Gamma^1_{\ell}(v) \). If it does, we set \( U^2_j(s) = B_1(x^s_{1,0} \cup \Gamma^2_{\ell}(c)) \) for \( i \leq j < s \) and return to the active mode. If \( U^1_{\ell} \) contains no witness we set \( U^2_i(s) = U^2_j(s_0) \) for \( j < s_0 \) and \( U^2_i(s) = \emptyset \) for \( s_0 \leq i < s \), and return to the active mode.

This concludes worker 2's construction.

Several observations: -Again we maintain consistency while trying
to realize any consistent type.

-If we shift into the waiting mode we will eventually shift out
of it.

-As before, once \( U^2_1, \ldots, U^2_{s-1} \) and \( K^3_{i,s} \) settle down, \( U^2_i \) will
settle down.

-At stage \( s \) worker 2 need only know about \( U^1_i, \ldots, U^1_{s-1} \). As these
are computed without reference to \( U^2_i(s) \), \( i < s \) we avoid circularity
in our use of the recursion theorem.

-Workers 1 and 2 have guaranteed that for \( i = 1,2 \) if \( \Gamma^i_j(v) \) is
consistent with the full diagram of \( M \), then it is realized.

**Worker 3:**

Worker 3 will be in either the "active" mode, the "waiting" mode
or the "passive" mode.

If worker 3 is active or waiting and worker 2 shifts from the
active to the waiting mode, worker 3 becomes passive.

When worker 2 becomes active again, worker 3 returns to the state
he was previously in.
Case 1: Worker 3 is passive.

Worker 3 sets $T_s^3 = T_{s-1}^3$.

Case 2: Worker 3 is active.

Worker 3 considers the first $\varphi_1^3$ which has not yet been considered and sets $T_s^3 = T_{s-1}^3 \land \varphi_1^3$ or $T_s^2 = T_{s-1}^2 \land \varphi_1^3$ to maintain consistency of $Th(N) \cup T_s^2 \cup T_s^3 \cup U_1 \cup \cdots \cup U_{s-1}^2$. If we add $\exists x \psi(x)$ we switch to the waiting mode to find a witness for $\psi(x)$.

Case 3: Worker 3 is waiting for a witness to $\psi(x)$, where $\exists x \psi(x)$ was added at stage $s_0$.

Worker 3 calculates $s_0$ s.t. $T_{s_0}^2 = B(T_{s_0}^3 \cup \psi(v))$. If $s \leq s_0$, let $T_s^3 = T_{s_0}^3$. If $s > s_0$, by our construction there is a witness $c$ to $T_{s_0}^2(v)$ in $U_{s_0}^2$. Let $T_s^3 = T_{s-1}^3 \land \psi(c)$.

This concludes worker 3's construction.

We observe that:
- The passive mode insures that worker 3 remains inactive during the period when worker 2 is not paying attention to its actions.
- As worker 2 always leaves the waiting mode, worker 3 will eventually leave the passive mode.
- Worker 3 will always return from the waiting mode to the active mode.
- Player 3 insures that $T^3$ is complete, consistent and Henkinized.

From $T^1$ we pass to $M$, the canonical model of $T^3$, effectively in $d'$. Thus $Diag(M) \equiv d'$. To see that $M$ is nonstandard, we need only observe that $\Gamma(v) = \{v \neq n : n \in \omega\}$ is consistent and thus realized in $M$. This concludes the proof of Theorem 3.5.
§5. S-SATURATION AND EFFECTIVE MODEL THEORY

In this section we state the basic results about recursively saturated structures which will be used in the following sections. The preliminary material in this section is due primarily to Wilmers [W1], [W2], and Knight and Nadel [KN1], [KN2]. We also prove the main theorem of recursive model theory that we will be using. Throughout we assume all structures are countable, and are not necessarily models of arithmetic.

Definition 5.1: Suppose $S \subseteq P(\omega)$. $M$ is S-saturated iff

i) every pure type realized in $M$ is recursive in some $s \in S$ and

ii) if $p(\bar{x}, \bar{y})$ is a pure type recursive in some $s \in S$, $\bar{m} \in M$ and $p(\bar{x}, \bar{m})$ is consistent, then $p(\bar{x}, \bar{m})$ is realized in $M$.

The following lemma summarizes the facts we will be using about S-saturation and Scott sets.

If $M$ is an L-structure $\text{Typ}(M)$ denotes the set of (codes of) pure types realized in $M$.

Lemma 5.2: i) If $M$ is S-saturated, then $\text{Typ}(M) \subseteq S$ and $\text{Th}(M) \in S$.

ii) $M$ is recursively saturated iff $M$ is S-saturated for some Scott set $S$.

iii) If $S$ is a countable Scott set and $T$ is a complete theory recursive in some $s \in S$, then $T$ has a countable S-saturated model.

iv) If $M$ is countable and S-saturated, then $M$ is determined up to isomorphism by $S$ and $\text{Th}(M)$.
Proof: i) is immediate from our definitions.

ii) Clearly if $M$ is $S$-saturated, then $M$ is recursively saturated. For the converse let $S = \{d \subseteq \omega: \text{if } \bar{m} \in M, \Gamma(v, \bar{m}) \text{ is consistent with } Th(M) \text{ and } \Gamma \text{ is recursive in } d, \text{ then } \Gamma(v, \bar{m}) \text{ is realized in } M\}$. Then $M$ is $S$-saturated and $S$ is a Scott set. See [KN1] for details.

iii) This is proved by a Henkin argument similar to the one used by Scott [Sc].

iv) Any two $S$-saturated models of a complete theory are $\omega$-homogeneous and realize the same types.

For many theories 5.2 iii) may be significantly strengthened.

**Definition 5.3:** $T$ is effectively perfect if for some $n \in \omega$ there is a recursive map $\phi: 2^\omega \to$ consistent formulas in $n$-variables s.t.

i) if $\sigma \subseteq \tau$, then $T \vdash \phi(\tau) \rightarrow \phi(\sigma)$ and

ii) $\phi(\sigma \wedge 0)$ and $\phi(\sigma \wedge 1)$ are incompatible.

**Examples:** i) If $T$ is a complete extension of Pressburger arithmetic in any suitable recursive language, then $T$ is effectively perfect.

The function $\phi$ may be given by $\phi(\alpha) = M[\varphi_n|v: \alpha (n) = 1] \land M[\varphi_n|v: \alpha (n) = 0]$.

ii) Macintyre [Mac1] has shown that, with the possible exception of some bizarre cases not known to exist, all theories of infinite fields which are not algebraically closed are effectively perfect.

iii) Wilmer [W2] and Knight-Nadel [K-N1] show that if $T$ is not atomic, then $T$ is $T$-effectively perfect.

iv) Knight-Nadel [K-N1] show that if $T$ has an infinite recursive sequence of independent formulas in $n$ free variables, then $T$ is effectively perfect.
Theorem 5.4: ([K-N1],[N],[W2]). If T is effectively perfect and
M ⊨ T is recursively saturated, then M is S-saturated for a unique
Scott set S.

Proof: Fix n and ϕ witnessing the perfection T. Let
m ∈ M^n. We say m codes f ∈ 2^ω iff M ⊧ ϕ(α)(m) for all α ⊆ f.
Let S = {X ⊆ ω: some m ∈ M^n codes the characteristic function
of X}.

Claim: S is a Scott set and M is S-saturated.

(We will show only that S is closed under Turing reducibility. The
rest of the proof is similar.) Suppose m codes f and f_1 ≤T f.
Consider the type Γ(ϕ) = {ϕ(β)(ϕ):β ⊆ f_1}. By 5.2 1) and 2) M must
realize Γ(ϕ) by some n. This n codes f_1. Lemma 5.2 also implies
M is S-saturated.

Suppose M is also S'-saturated. Then S' ⊇ S. Let s' ∈ S'.
Coding arguments similar to the above show s' is coded by some
m ∈ M^n. Thus S' = S. //

Knight-Nadel show that the conclusion of 5.4 also holds if T
has pure types of every degree. Some condition on T is evidently
necessary as any countable saturated model is S-saturated for all suf-
ficiently large S.

We devote the remainder of this section to the effective model
theory needed to study recursively saturated models of arithmetic. In
what follows let L be a fixed recursive language. If ω L = (ω,...)
is an L-structure with universe ω, Diag(L) denotes the atomic diagram
of, and Diag*(L) denotes the full diagram of L. We let deg(L) and
\( \text{deg}^*(\mathcal{A}) \) denote the Turing degrees of \( \text{Diag}(\mathcal{A}) \) and \( \text{Diag}^*(\mathcal{A}) \) respectively. If \( \mathcal{A} \) is a structure, \( \text{Typ}(\mathcal{A}) \) denotes the set of pure types realized in \( \mathcal{A} \).

**Lemma 5.5:** If \( \mathcal{A} = (\omega, \ldots) \) is an \( L \)-structure, then there is a \( \text{Diag}^*(\mathcal{A}) \)-enumeration of \( \text{Typ}(\mathcal{A}) \).

**Proof:** Let \( \sigma: \omega \to \omega \) and \( \tau: \omega \to L \)-formulas be recursive bijections. Define \( E \subseteq \omega^2 \) by \((n,m) \in E \) iff \( \mathcal{A} \models \varphi(i_1, \ldots, i_k) \) where \( \sigma(n) = \langle i_1, \ldots, i_k \rangle \) and \( \tau(m) = \varphi(\vec{v}) \).

Our main goal is to prove the converse of 5.5 for recursively saturated structures. This will require ideas of Goncharov [Go] and Peretyat'kin [P] which are

**Definition 5.6:** Let \( S \) be a set of types. A \( d \)-enumeration \( E \) of \( S \) has the \textit{d-effective extension property} iff there is a \( d \)-recursive \( g: \omega^2 \to \omega \) s.t. if \( p(x_0, \ldots, x_{n-1}) \in S \), say \( p = \{ \psi: (m, r, \vec{y}) \in E \} \), and \( \varphi(x_0, \ldots, x_n) \) is consistent with \( p \), then \( \{ \psi: (g(m, r, \vec{y}), \vec{r}, \vec{y}) \in E \} \) is an extension of \( p \) in the variables \( x_0, \ldots, x_n \) containing \( \varphi(x_0, \ldots, x_n) \).

**Theorem 5.7:** (Goncharov [Go] and Peretyat'Kin [P]). Let \( \mathcal{A} \) be \( \omega \)-homogeneous. Let \( E \subseteq \omega^2 \) be a \( d \)-enumeration of \( \text{Typ}(\mathcal{A}) \) with the \( d \)-effective extension property. Then there is \( \mathcal{B} = (\omega, \ldots) \) s.t. \( \mathcal{B} \equiv \mathcal{A} \) and \( \text{Diag}^*(\mathcal{B}) \leq^d \mathcal{B} \).

We will show that the hypothesis of 5.7 holds reasonably often.

**Fix** \( T \) **a complete** \( L \)-**theory**.

**Definition 5.8:** A set \( S \) of types over \( T \) is \textit{Turing closed} iff for any \( p \in S \) if \( q \) is a type over \( T \) and \( q \leq_T p \), then \( q \in S \).
Let \( X \subseteq P(\omega) \) with \( T \) coded by some \( x \in X \). Let \( \text{Typ}(X) \) be the set of complete types which are coded in \( X \). \( \text{Typ}^n(X) \) will denote the complete \( n \)-types coded in \( X \).

**Lemma 5.9:** If \( \text{Typ}(X) \) is Turing closed and \( E \) is an enumeration of \( X \), then \( \text{Typ}(X) \) has an enumeration recursive in \( E \).

**Proof:** For notational simplicity we will consider only \( \text{Typ}^1(X) \). We dovetail over \( \omega^2 \) to decide larger and larger parts of each \( E_m \). We say \( m \) is active at stage \( s \) of our construction iff

1. \( m \) is active at each stage \( s' < s \).
2. \( \{ n \leq s : n \in E_m \} \) codes a set of formulas in one variable consistent with \( T \), and

3. there is no formula \( \varphi(v) \) s.t. \( \gamma \varphi(v) \gamma \leq s \), \( \gamma \varphi(v) \gamma \leq s \) and neither \( \gamma \varphi(v) \gamma \) or \( \gamma \varphi(v) \gamma \) are in \( E_m \).

These conditions are all recursive in \( T \) and as \( T \) is coded in \( X \), they can be answered recursively in \( E \).

If \( m \) is not active at stage \( s \), there is a greatest \( t < s \) such that \( m \) was active at stage \( t \). Thus \( \Gamma = \{ \varphi(v) : \gamma \varphi \gamma \leq t \text{ and } \gamma \varphi \gamma \in E_m \} \cup \{ \varphi(v) : \gamma \varphi \gamma \leq t \text{ and } \gamma \varphi \gamma \notin E_m \} \) codes a consistent set of formulas. We may now uniformly, and effectively in \( T \), complete \( \Gamma \) by a Henkin process.

We define \( E^* \subseteq \omega^2 \) as follows: \((m,n) \in E^* \) iff

1. \( m \) is active at stage \( n \) and \((m,n) \in E \), or
2. the Henkin process puts the formula with Godel code \( n \) into the extension of \( \Gamma \).

Clearly \( E^* \leq_T E^* \). If \( p \) is a type in one variable coded in \( X \), and \( \varphi \in p \) if \( \gamma \varphi \gamma \in E_m \), then \( \varphi \in p \) iff \( \gamma \varphi \gamma \in E_m \). Also if
q = \{ \varphi \in E_n \mid q_n \in E_n \}\), then either (i) \( n \) is active at every stage \( s \), in which case \( q \) is in \( \text{Typ}^1(X) \) or (ii) \( n \) is deactivated at some stage \( s \). In this case \( q \leq_T \) and since \( \text{Typ}^1(X) \) is Turing closed, \( q \in \text{Typ}^1(X) \). Thus \( E^* \) is an enumeration of \( \text{Typ}^1(X) \).

Similar proofs work for each \( \text{Typ}^n(X) \). Putting these enumerations together gives an enumeration of \( \text{Typ}(X) \).

Lemma 5.10: Let \( T \) be a complete theory. There is a uniform recursive operator \( Y \) such that if \( p \) is a type over \( T \) in the variables \( x_0, \ldots, x_{n-1} \) and \( \varphi(x_0, \ldots, x_n) \) is a formula consistent with \( p \), then \( Y(p, \varphi) \) is a type in the variables \( x_0, \ldots, x_n \) s.t. \( Y(p, \varphi) \geq_T p \), \( \varphi \in Y(p, \varphi) \) and \( Y(p, \varphi) \leq_T p \).

Proof: \( Y \) is just the usual Henkin procedure for finding a complete extension of \( p \) containing \( \varphi \).

Theorem 5.11: If \( S \subseteq P(\omega) \) is closed under Turing reducibility, \( E \) is an enumeration of \( S \) and \( E \leq_T d \), then \( \text{Typ}(S) \) has an enumeration \( G \) with the \( d \)-effective extension property.

Proof: We inductively define a set \( I \) of indices by the following rules:

i) if \( n \in \omega \), then \( n \in I \), and

ii) if \( i \in I \) and \( \varphi(x_0, \ldots, x_n) \) is a formula, then \( <i, \varphi> \in I \).

Let \( E^* \subseteq \omega^2 \) be the enumeration of \( \text{Typ}(X) \) given by 5.9. Let \( \sigma : \omega \to I \) be a recursive bijection. We define \( G \subseteq \omega^2 \) inductively on indices as follows:

i) if \( \sigma(m) \in \omega \), \( G_m = E^* \),

ii) if \( \sigma(m) = <i, \varphi(x_0, \ldots, x_n)> \), \( \sigma(m_0) = i \) and \( q = \{ \theta : \theta^n \in G_{m_0} \} \)
is a type in the variables \( x_0, \ldots, x_{n-1} \) consistent with \( \varphi(x_0, \ldots, x_n) \),
then \( G_m \) codes \( \forall(q, \varphi(x_0, \ldots, x_n)) \),
\[ \text{iii) otherwise } G_m = E_0^* \.

Clearly \( G \) is an enumeration of \( \text{Typ}(X) \). Let \( h: \omega^2 \to \omega \) be
defined by \( h(m, \varphi) = \sigma^{-1}(< \sigma(m), \varphi>) \). \( h \) witnesses the \( d \)-effective
extension property.

We may now state our partial converse to 5.5.

**Theorem 5.12:** If \( \mathcal{A} \) is \( \omega \)-homogeneous, \( S \) is the set of degrees of
pure types realized in \( \mathcal{A} \), \( \text{Typ}(A) \) is Turing closed and \( E \) is an
enumeration of \( S \), then there is \( \mathcal{B} \cong \mathcal{A} \) with \( \text{Diag}^* \mathcal{B} \leq_E \).

**Proof:** Clear from 5.7 and 5.11.

Goncharov [Go] and Peretyat'kin [F] have examples which show
that we cannot omit the assumption that \( \text{Typ}(M) \) is Turing closed. We
will be using only the following corollary:

**Corollary 5.13:** If \( \mathcal{A} \) is \( S \)-saturated and \( E \) is an enumeration of \( S \),
then there is \( \mathcal{B} \cong \mathcal{A} \) with \( \text{Diag}^* \mathcal{B} \leq_E \).
§6. APPLICATIONS TO ARITHMETIC

In this section we will give applications of 5.13 to the study of structures arising in arithmetic. We begin by fixing some notation.
If $L$ is a recursive language and $\mathfrak{a} = (\omega, \ldots)$ is an $L$-structure, then $\text{Diag}(\mathfrak{a})$ denotes atomic diagram of $\mathfrak{a}$ and $\text{deg}(\mathfrak{a})$ is the Turing degree of $\text{Diag}(\mathfrak{a})$, $D(\mathfrak{a}) = \{\text{deg}(\mathfrak{b}) : \mathfrak{a} \cong \mathfrak{b}\}$, and $D^*(\mathfrak{a}) = \{\text{deg}^*(\mathfrak{b}) : \mathfrak{a} \cong \mathfrak{b}\}$. If $S \subseteq P(\omega)$ is countable, $D(S) = \{d : \text{there is a } d\text{-enumeration of } S\}$.

a) Pressburger arithmetic.

Let $T$ be a complete theory in a language extending $L = \{+, 0, 1\}$ such that $T \supseteq \text{Th}(\omega, +, 0, 1)$. For $M \models T$, $M_+$ denotes the reduct of $M$ to $L$. If $a \in M$, we define the real coded by a $r(a) = \{n \in \omega : M \models p_n(a)\}$ and let $\text{Re}(M) = \{r(a) : a \in M\}$. We state several facts about $\text{Re}(M)$ which were proved in §1 for $M \models \text{PA}$.

Lemma 6.1: If $a \in M$, $r(a) \leq_\text{deg} \text{deg}(M)$. Moreover, there is a $\text{Diag}(M)$-enumeration of $\text{Re}(M)$.

Proof: The proofs of 1.8 and 1.11 work for any $M \models \text{Th}(\omega, +)$. //

We recall from Section 5 that $T$ is effectively perfect. Thus by 5.4, if $M \models T$ is recursively saturated, then $M$ is $S$-saturated for a unique Scott set $S$. We will often refer to such an $S$ as the Scott set of $M$.

Lemma 6.2: If $M \models T$ is recursively saturated and $S$ is the Scott set of $M$, then $S = \text{Re}(M)$.

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Proof: We need only show \( M \) is \( \text{Re}(M) \)-saturated. Let \( \bar{a} \in M \) and let \( \Gamma(v) = \{ pr_\varphi \gamma | v \leftrightarrow \varphi(\bar{a}) : \varphi(\bar{v}) \text{ on } L\text{-formula} \} \). As \( M_+ \) is recursively saturated, \( \Gamma \) is realized by some element of \( M \). Thus every pure type realized is recursive in some element of \( \text{Re}(M) \). Let \( \Gamma(\bar{v}, \bar{w}) \) be a type coded in \( \text{Re}(M) \), i.e., there is \( a \in M \) s.t. \( \varphi(\bar{v}, \bar{w}) \in \Gamma \iff pr_\varphi \gamma | a \). Then for any \( \bar{m} \in M \), \( M \) must realize \( \{ \varphi(\bar{v}, \bar{m}) \leftrightarrow pr_\varphi \gamma | a : \varphi \text{ on } L\text{-formula} \} \). Hence \( M \) is \( \text{Re}(M) \)-saturated. //

Lemma 6.3: If \( M \models T \), \( \bar{a} \in D(M) \), and \( \bar{e} \geq \bar{a} \), then \( \bar{e} \in D(M) \). Similarly, if \( \bar{a} \in D^*(M) \), \( \bar{e} \in D^*(M) \).

Proof: The proof of 2.2 works for \( M \models T \). //

Corollary 6.4: If \( M \models T \) is S-saturated, then \( D(M) = D^*(M) = D(S) \).

Proof: Corollary 5.13 shows \( D^*(M) \) is dense in \( D(S) \). Clearly \( D(M) \) is dense in \( D^*(M) \). As \( S = \text{Re}(M) \), \( D(S) \) is dense in \( D^*(M) \). Lemmas 6.3 and 1.12 show that \( D(M) = D^*(M) = D(S) \). //

We note that the conclusion \( D(M) = D^*(M) \) uses both recursive saturation and the coding power of arithmetic. In general, if we know \( \bar{a} \in D(M) \) we can only conclude \( \bar{a}^{(\omega)} \in D^*(M) \). Harrington's theorem (3.2) shows that this is all we can expect. The following proposition gives an example showing that \( D(M) = D^*(M) \) is not a general property of recursively saturated models.

Proposition 6.5: There is a complete undecidable \( \omega_1 \)-categorical theory \( T \) with a recursively presented \( \omega \)-saturated model \( \mathcal{M} \). (Thus \( \bar{a} \in D(\mathcal{M}) \), but \( \bar{a} \notin D^*(\mathcal{M}) \).)
Proof: If \( \{n\}(n)^+ \), let \( u_n = 2s \) where \( s\{n\}(n) \) halts exactly at step s. For all n let

\[
U_n = \begin{cases} 
\{u_n\} & \text{if } \{n\}(n)^+ \\
\emptyset & \text{otherwise.} 
\end{cases}
\]

Let \( \mathcal{A} = (\omega, U_n : n \in \omega) \) and let \( L = \{U_i : i \in \omega\} \). Th(\( \mathcal{A} \)) is undecidable as \( \{n\}(n)^+ \) iff \( \exists \forall U_n (v) \in \text{Th}(\mathcal{A}) \). The predicate \( "m \in U_n" \) is recursive uniformly in m and n so \( \mathcal{A} \) is recursively presented.

Suppose \( \mathcal{A} \models \text{Th}(\mathcal{A}) \). Let \( X = \{b \in \mathcal{A} : \exists \forall U_n (v) \in \mathcal{A} \} \). If \( \mathcal{E} \models \text{Th}(\mathcal{A}) \) \( X^\mathcal{E} \cong X^\mathcal{A} \). Thus the isomorphism type of \( X^\mathcal{E} \) is determined by \( X^\mathcal{A} \). As no relations hold on \( X^\mathcal{A} \), the isomorphism type of \( X^\mathcal{E} \) is determined by \( |X^\mathcal{A}| = \Omega_0 \), \( \Omega \) is \( \omega \)-saturated. //

b) Skolem arithmetic.

Let T be a complete extension of \( \text{Th}(\omega, \cdot) \) is some recursive language extending \( \{\cdot, 0, 1\} \). Let \( M \models T \) and let \( b \in M \) be an infinite power of 2. If \( a \in M \) and \( a|b \) we define \( r_b^*(a) = \{n: \exists x p_n = a\} \). Let \( \text{Re}_b(M) \) denote \( \{r_b^*(a): a|b\} \).

Lemma 6.6: If \( g \in D(M) \), then for all \( a|b \), \( r_b^*(a) \leq_T d \), uniformly in \( a,b \). Further, there is a \( d \)-enumeration of \( \text{Re}_b(M) \).

Proof: Clearly \( r_b^*(a) \) is r.e. in \( d \), so we need only show \( \{n: \exists x p_n = a\} \) is r.e. in \( d \). As \( a \) is a power of 2 there is an \( m \) s.t. \( a = 2^m \). Thus \( \exists x p_n = a \iff \exists m p_n^m \). Thus \( \exists x p_n = a \iff \exists x p_n^m \). Hence, \( \{n: \exists x p_n = a\} \) is r.e. in \( d \). To obtain a \( d \)-enumeration of \( \text{Re}_b(M) \) we begin a list \( a_0, a_1, a_2, \ldots \) of all elements of \( M \) which divide \( b \). Our enumeration is \( E = \{(n,m) : m \in r_b^*(a_n)\} \). //
Lemma 6.7: T is effectively rich and if $M \models T$ is S-saturated then $S = \text{Re}^*_b(M)$, where $b$ is any nonstandard power of 2.

Proof: The function $\varphi(a) = \models \{ \exists x \ x^n = v : c(n) = 1 \} \wedge \models \{ \exists x \ x^n = v : c(n) = 0 \}$ shows that T is effectively perfect. The second part of the proof is done as in 6.2. //

In particular, for recursively saturated $M \models T$, $\text{Re}^*_b(M)$ does not depend on our choice of $b$, so we may denote $S$ as $\text{Re}^*_M$.

Corollary 6.8: If $M \models T$ is S-saturated and $\mathcal{A} \in D(M)$, then $\mathcal{A} \in D(S)$.

Proof: Let $b$ be an infinite power of 2. Let $m_0, m_1, m_2 \ldots$ be a listing of $M$. We begin dovetailing searches for $x$ such that $xm_1 = b$ and list as $n_0, n_1, \ldots$ all $m_i$ for which such an $x$ is found. This can be done d-effectively. Let $E = \{(i, j) : j \in T^*_b(n_i)\}$. $E \subseteq T^d$, since the decision $j \in T^*_b(n_i)$ is d-recursive uniformly in $i$. By 6.7 $E$ is an enumeration of $S$. By 1.12 $\mathcal{A} \in D(S)$. //

In order to classify degrees coding recursively saturated models of $T$ we need a padding lemma analogous to 6.3. This was noticed by Julia Knight.

Lemma 6.9: Let $M \models \text{Th}(\omega, \cdot)$. Suppose $a, b \in M$ s.t. $a$ is an infinite power of 2 and $b$ is an infinite power of 3, Let $\mathcal{A} \in D(M)$, if $e \geq d$, then $\mathcal{A} \in D(M)$.

Proof: Let $M = (\omega, \emptyset)$ with $\emptyset \leq_T d$. Pick $a$ and $b$ out of $M$. We can find a function $f : \omega \to \omega$ a bijection such that

1) $f \leq_T e$, 

2) if $n \in e$, $M \models f(2n) \upharpoonright a$, 

3) if $n \notin e$, $M \models f(2n) \upharpoonright b$. 
Define \( w \) by \( n \vDash w = p \) iff \( f(n) \vDash f(m) = f(p) \). \( M \vDash (\omega, w) \), 
\( w \leq^T e \) and as \( n \in e \) iff \( \exists m \ vdash 2n = a \) and \( n \notin e \) iff \( \exists m \ vdash 2n = b \), 
\( e \leq^T w \).

Knight has shown that we do not really need \( a \) and \( b \) to prove 6.9, but as we are only considering recursively saturated structures, this assumption is harmless. We may now prove the desired classification result.

**Corollary 6.10:** If \( M \models T \) is S-saturated, then \( D(M) = D^*(M) = D(S) \).

**Proof:** As in 6.4.

**c)** Subsystems of PA.

\( \Sigma_n \) will denote the subsystem of PA obtained by restricting
the induction schema to \( \Sigma_n \) formulas. EXP will denote \( \Sigma_0^* + \forall x, y \exists z \ x^y = z \), where this is suitably formalized. It can be shown that
\( \Sigma_1^2 \supseteq \text{EXP} \supseteq \Sigma_0^2 \supseteq \text{Th}(\omega, +) U \text{Th}(\omega, \cdot)) \).

If \( M \) is a \( \{+, \cdot, 0, 1\} \), \( M_+ \) will denote the reduct of \( M \) to
\( \{+, 0, 1\} \) and \( M_\cdot \) will denote the reduct of \( M \) to \( \{\cdot, 0, 1\} \). Putting
together 6.2, 6.4, 6.7, and 6.10, we get the following corollary.

**Corollary 6.11:** If \( M \vDash \Sigma_0 \) is S-saturated, then \( S = \text{Re}(M) = \text{Re}^*(M) \)
and \( D(M) = D^*(M) = D(S) = D(M_\cdot) = D^*(M_\cdot) = D(M_+) = D^*(M_+) \).

Some of our results may be extended to models which are not recursively saturated. We will require the following results of
Cegielski-McAloon and Wilmers.
Theorem 6.12: (Cegielski-McAloon-Wilms [C-M-W]).

1) Let $M \models I\Sigma_0$, then $M^+_\omega$ is recursively saturated.
2) Let $M \models EXP$, then $M^*_0$ is recursively saturated.

Theorem 6.14 extends results of Lipshitz and Nadel [L-N] for PA. Cegielski, McAloon and Wilms point out that EXP cannot be weakened to $I\Sigma_0$ in 2).

Lemma 6.13: Let $M \models I\Sigma_0$ be nonstandard and let $b \in M$ be an infinite power of 2. Then $Re(M) = Re^*(M)$.

Proof: Let $r(a) \in Re(M)$. For $n \in \omega \models a \models n(p_m)^n \models v \Longleftrightarrow (\exists z < v \exists z^m = v)$). By $\Sigma_0$-overspill this must also hold for some nonstandard $N$. If $c$ witnesses this, then $r(a) = r^*(c)$. Thus, $Re(M) \subseteq Re^*(M)$. The proof of the reverse inclusion is similar. //

Corollary 6.14: If $M \models I\Sigma_0$ is nonstandard, then $D(M^+_\omega) = D^*(M^+_\omega) = D(Re(M))$.

Proof: Clear from 6.12, 6.13, and 6.4. //

Corollary 6.15: If $M \models EXP$ is nonstandard, then $D(M^*_0) = D^*(M^*_0) = D(Re(M))$.

Proof: Clear from 6.12, 6.13 and 6.11. //

d) Initial segments.

Let $M \models I\Sigma_0$. Let $a$ be a nonstandard element of $M$. We define a structure $[0,a]$ with universe $\{x \in M : x \leq a\}$ and functions + and · defined by:
\[ x + y = \begin{cases} x + y & \text{if } M \models x + y \leq a \\ a & \text{if } M \models x + y > a \end{cases} \]

\[ x \cdot y = \begin{cases} x \cdot y & \text{if } M \models x \cdot y \leq a \\ a & \text{if } M \models x \cdot y > a. \end{cases} \]

If \( b \in [0,a] \), we define \( r^*_a(b) = \{ n: p_n | b \} \).

**Lemma 6.16:** \( \Re(M) = \{ r^*_a(b): b \leq a \} \).

**Proof:** This follows from the fact that reals are coded by arbitrarily small nonstandard numbers, which we proved in 1.31. //

**Corollary 6.17:** \( D([0,a]) \subseteq D(\Re(M)) \).

**Lemma 6.18:** \( D([0,a]) \) and \( D^*([0,a]) \) are closed upward in the Turing degrees.

**Proof:** Similar to 2.2. //

**Corollary 6.19:** If \( M \models I \Sigma_0 \) and \([0,a]\) is \( \Re(M) \)-saturated, then \( D([0,a]) = D^*([0,a]) = D(\Re(M)) \).

The following result shows the hypothesis of 6.19 holds reasonably often.

**Lemma 6.20:** (Lessan [Le]). If \( M \models EXP \) and \( a \in M \) is nonstandard, then \([0,a]\) is recursively saturated.

We will show in the appendix that the hypothesis of 6.20 cannot be weakened to \( M \models I \Sigma_0 \).

**Corollary 6.21:** If \( M \models EXP \) and \( a,b \in M \) are nonstandard then \( D([0,a]) = D([0,b]) = D(\Re(M)) \).

Stronger results are possible if we look at \([0,a]_+\), the reduct of \([0,a]\) to addition. Clearly, \(\text{Re}([0,a]) = \text{Re}([0,a]_+)\).

**Lemma 6.22:** (Cegielski-McAloon-Wilmers [C-M-W]). If \(M \models \text{IE}_0\), then \([0,a]_+\) is recursively saturated.

**Corollary 6.23:** If \(M \models \text{IE}_0\) and \(a, b \in M\) are nonstandard, then \(D([0,a]_+) = D([0,b]_+) = D(\text{Re}(M))\).

e) Residue Fields.

Suppose \(M \models \text{PA}\) and \(p \in M\) is an infinite prime. Then \(M/p\) is a field of characteristic 0.

**Lemma 6.24:** (Macintyre [Mac2]).

1) \(M/p\) is a pseudofinite field (in the sense of \(\text{Ax}[A]\)), \(M/p\) is recursively saturated and every recursively saturated pseudofinite field is of the form \(M/p\).

2) If \(T\) is a complete theory of pseudofinite fields, then \(T\) is effectively perfect.

Proof: 1) See Macintyre [Mac2].

2) Let \(\varphi\) be the map \(\varphi(a) = M \{\exists x(v+n = x^2) : a(n) = 1\} \land M\{\exists x(v+n = x^2) : a(n) = 0\}\). Macintyre [Ma1] shows that \(\varphi\) witnesses the perfection of \(T\). //

Let \(T\) be a complete theory of pseudofinite fields of characteristic 0. Suppose \(K \models T\). If \(\delta \in K\) is a nonsquare then \(\exists x \, v+n = x^2\) iff \(\exists x \, (v+n = \delta \cdot x^2)\). Suppose \(a \in K\). Let \(r_K(a) = \{n \in \omega : \exists x(a+n = x^2)\}\. 
The above remark shows that $r_K(a)$ is recursive in $\deg(K)$ uniformly in $a$. Let $\text{Re}(K) = \{r_K(a): a \in K\}$. The following lemma is then clear.

**Lemma 6.25:** 1) There is a $\deg(K)$ enumeration of $\text{Re}(K)$.

2) If $K$ is $S$-saturated, then $S = \text{Re}(K)$.

**Lemma 6.26:** Suppose $K = M/p$ and $K$ is $S$-saturated. Then $S = \text{Re}(M)$.

**Proof:** As $M/p$ is definable in $M$, $\text{Re}(K) \subseteq \text{Re}(M)$. Thus by 6.25 $S \subseteq \text{Re}(M)$.

Suppose $a \in M$. Consider the type $\{\exists x \forall n = x^n (\text{mod } p) \iff p/\omega \exists n \in \omega\}$. This is recursive and of bounded complexity. Thus, by 1.16, it is realized in $M$. Thus $r(a) \in \text{Re}(M/p)$. So $S = \text{Re}(M/p)$.

As usual we must prove a padding lemma for pseudofinite fields.

**Lemma 6.27:** If $d \in D(K)$ and $e \geq d$, then $\xi \in D(K)$. (Similarly $D^*(K)$ is closed upwards.)

**Proof:** Let $K = (\omega, \oplus_p, \odot_p)$ have degree $d$. We can find $f: \omega \to \omega$ a bijection such that $f$ is recursive in $e$ and $f(n)$ is a square iff $n \in e$. Let $K'$ be the field induced by $f$. Then $K' \cong K$ and $\deg(K') = \xi$.

**Corollary 6.28:** Let $M \models PA$ and let $p \in M$ be a nonstandard prime, then $D(M/p) = D^*(M/p) = D(\text{Re}(M))$. In particular, if $b$ is also a nonstandard prime, $D([0,a]) = D([0,b])$.

**Proof:** As in 6.4.

Let $M/p = (\omega, \oplus_p, \odot_p)$. $(\omega, \oplus_p)$ is a divisible abelian group. In fact, $(\omega, \oplus_p) \cong Q(\omega)$. It follows from the next proposition that
$D((\omega,\Theta))$ is the set of all Turing degrees.

**Proposition 6.29:** $Q^{(\omega)}$ has presentations of every degree.

**Proof:** Let $b$ be a nonzero element of $Q^{(\omega)}$ and write $Q^{(\omega)} = \langle b \rangle \Theta D$, where $\langle b \rangle$ is the span of $b$ and $D \cong Q^{(\omega)}$. $D$ has a recursive presentation $(\omega, \oplus)$. Let $g: D \rightarrow \omega$ be a recursive isomorphism. Any $x \in Q^{(\omega)}$ has a unique representation $x = rb + d$, where $d \in D$ and $r \in Q$.

Let $h: Q \rightarrow \omega$ be a bijection. We define $f: Q^{(\omega)} \rightarrow \omega$ as follows. $f(rb + d) = \langle h(r), g(d) \rangle$, where $\langle , \rangle$ is some fixed recursive pairing function. For $i = 0, 1, 2$ let $f(r_i b = d_i) = n_i$. Then $Q^{(\omega)} \models n_0 + n_1 = n_2$ iff $d_0 + d_1 = d_2$ and $r_0 + r_1 = r_2$. Thus if $(\omega, \Theta)$ is the group induced by $f$, $\Theta$ is recursive in $h$.

On the other hand,

$$h(m/n) = k \iff (\omega, \Theta) \models m/n \models f(b) = \langle k, g(0) \rangle$$

$$\iff (\omega, \Theta) \models mf(b) = n \langle k, g(0) \rangle.$$

Thus, given the diagram of $(\omega, \Theta)$ we can compute $h$. As $h$ may be chosen of arbitrary complexity, $Q^{(\omega)}$ has presentations in every degree.

The situation for $(\omega, \Theta_p)$ is more delicate. Macintyre has shown that in any $M \models PA$ there are nonstandard primes s.t. $D(\omega, \Theta_p)$ is the set of all Turing degrees. On the other hand there are primes such that $D(\omega, \Theta_p)$ is not the set of all Turing degrees. Macintyre's precise result is that $D(\omega, \Theta_p) = \{ d : d \geq \text{the degree of } \Theta_p \}$. //
The results of this section are true if instead of having $M \models \text{PA}$ we have $M \models \Sigma_1$. It is reasonable to conjecture that $\exp$ is enough. As for $\Sigma_0$, it is not even known that $M/p$ cannot be algebraically closed!
§7. CLASSIFICATION RESULTS

Let $T$ be a complete extension of $PA$. In this section we classify degrees coding structures related to models of $T$. We also discuss the relationship between enumerations of Scott sets and effective enumerations of Scott sets. The easiest result is the following.

**Proposition 7.1:** $d$ codes a recursively saturated model of $T$ iff $d$ enumerates a Scott set containing $T$.

**Proof:** Clear from 5.2 i and 5.13. //

**Corollary 7.2:** $d$ codes a recursively saturated model of $T$ iff $d$ enumerates a Scott set containing $0^{(\omega)}$.

Our main tool is the following theorem of Knight [Kl].

**Theorem 7.3:** (Knight [Kl]). Let $S$ be a countable Scott set. $S$ occurs as $\text{Re}(M)$ for some $M \models T$ iff for all $n \in \omega$, $T \cap \Sigma_n \in S$.

**Corollary 7.4:** (Knight [Kl]). $M \models T(\omega,+)$ is expandable to a model of $T$ iff $M$ is recursively saturated and for all $n \in \omega$, $T \cap \Sigma_n \in \text{Re}(M)$.

**Proof:** Let $N \models T$ with $\text{Re}(N) = \text{Re}(M)$. Then $N_+ \models \text{Re}(M)$-saturated, so $N_+ \cong M$ by 5.2 iv). //

**Corollary 7.5:** $d$ codes the additive structure of a model of $T$ iff $d$ enumerates a Scott set $S$ containing all $T \cap \Sigma_n$.

**Proof:** ($\Rightarrow$) is clear.

($\Leftarrow$) Let $d$ enumerate $S$. By 5.13 we can find $N \models \text{Th}(\omega,+)$ with $\text{Re}(N) = S$ such that $N$ is recursively saturated and $\text{deg}(N) = \omega$. 53
But if \( M \models T \) with \( \text{Re}(M) = S \), then \( M_+ \cong N \). Thus \( d \) codes the additive structure of a model of \( T \).  

Similar results hold for \( \text{Th}(\omega, \cdot) \).

**Corollary 7.6:**  
i) Let \( M \models \text{Th}(\omega, \cdot) \). \( M \) is expandable to a model of \( T \) iff \( M \) is recursively saturated and for each \( n \in \omega \), \( T \cap \Sigma_n \subseteq \text{Re}(M) \).

ii) \( d \) codes the additive structure of a model of \( T \) iff \( d \) enumerates a Scott set \( S \) containing all \( T \cap \Sigma_n \).

**Proof:** As in 7.4 and 7.5.

Using the following result of Friedman's we can classify degrees coding initial segments and residue fields.

**Theorem 7.7:** Let \( M \) and \( N \) be countable models of \( \text{PA} \). \( M \) can be embedded as an initial segment of \( N \) iff \( \text{Re}(M) = \text{Re}(N) \) and \( \text{Th}_{\Sigma_1}(M) \subseteq \text{Th}_{\Sigma_1}(N) \).

**Proof:** See [Sm2].

**Corollary 7.8:** \( d \) codes \([0,a]\) where \( a \in M \) is a nonstandard model of \( T \) iff \( d \) enumerates a Scott set containing each \( T \cap \Sigma_n \).

**Proof:** (\( \Rightarrow \)) Clear.

(\( \Leftarrow \)) Let \( d \) enumerate \( S \). Let \( T^* \in S \) be a completion of \( \text{PA} \cup (T \cap \Sigma_1^0) \), and let \( N \models T^* \) be recursively saturated with \( \text{deg}(N) = d \). By 6.7 there is \( M \models N \) such that \( M \models T \). Thus, there is \( a \in M \) such that \( \text{deg}([0,a]) \leq d \). By 6.18, \( d \in D([0,a]) \).

**Corollary 7.9:** \( d \) codes \( M/p \) where \( M \models T \) and \( p \in M \) is a nonstandard prime iff \( d \) enumerates a Scott set containing all \( T \cap \Sigma_n \).
Proof: As in 7.8.

The next proposition is the surprising result that effectively enumerating a Scott set is no stronger than enumerating it.

**Proposition 7.10:** If \( d \) enumerates a Scott set \( S \), then \( d \) effectively enumerates \( S \).

**Proof:** Let \( T \in S \) be a complete extension of \( PA \). By 5.13, there is a recursively saturated \( N \models T \) such that \( Re(N) = S \) and \( \text{deg}(N) = d \). By 1.14, there is a \( d \) effective enumeration of \( S \). //

7.10 has several corollaries which were hinted at earlier. The most important is the classification of degrees coding models of \( \text{Th}(\mathbb{N}) \).

**Corollary 7.11:** Let \( S \) be a Scott set containing the arithmetical sets. There is \( M \models \text{Th}(\mathbb{N}) \) with \( Re(M) = S \) and \( \text{deg}(M) = d \) iff \( d \) enumerates \( S \).

We can strengthen one of the Knight-Lachlan-Soare results using 7.11 and the following theorem of Hodes [Ho].

**Theorem 7.12:** (Hodes [Ho]). There is an enumeration \( E \) of the arithmetical sets such that \( E'' \equiv^T_\omega \).

**Corollary 7.13:** There is \( M \models \text{Th}(\mathbb{N}) \) s.t. \( Re(M) \) is the set of arithmetical sets and \( \text{deg}(M)'' \equiv^T_\omega \).

7.10 may also be used to strengthen 1.27 and 2.1.

**Corollary 7.14:** The following are equivalent for any Scott set \( S \).

i) There is a \( d \)-effective enumeration of \( S \).

ii) There is a \( d \) enumeration of \( S \).
iii) There is a complete extension $T$ of PA s.t. $T \equiv d$
and $\text{Rep}(T) = S$.

iv) For any $T \in S$ a complete extension, there is a recursively saturated $M \models T$ with $\text{Re}(M) = S$ and $\text{deg}(M) = \delta$.

**Corollary 7.15:** $d$ enumerates a Scott set iff $d$ separates a pair of effectively inseparable r.e. sets.
§8. DEGREES CODING MODELS OF PT

In this section we shift gears and examine degrees coding models of PT, a theory introduced by DeMillo and Lipton to study the polynomial time hierarchy.

**Definition 8.1:** Let $L$ be the usual language of arithmetic augmented by function symbols for all polynomial time computable functions and relation symbols for all polynomial time decidable relations.

Let $PT$ be the set of all $\exists \forall$ sentences true in the standard model.

In [D–L], DeMillo and Lipton relate classical questions about the polynomial time hierarchy to questions of provability in $PT$. For example: Suppose $S \in NP \cap coNP$, then there are polynomial time recognizable predicates $A(x,y)$ and $B(x,y)$ such that $x \in S$ iff $\exists y A(x,y)$ and $x \notin S$ iff $\exists y B(x,y)$. We say $A$ and $B$ represent $S$. Let $\Delta_S(A,B)$ be the sentence $\forall x (\exists y A(x,y) \vee \exists z B(x,z))$.

**Theorem 8.2:** (DeMillo–Lipton [D–L]). Let $S \in NP \cap coNP$. Then $S \in P$ iff $PT \models \Delta_S(A,B)$ for some $A,B$ representing $S$.

**Proof:**

(\Rightarrow) If $S \in P$, there are $A(x)$ and $B(x)$ in $L$ such that $x \in S$ iff $A(x)$ and $x \notin S$ if $B(x)$. Clearly $PT \models \forall x (\exists y A(x) \vee \exists z B(x))$.

(\Leftarrow) Suppose $PT \models \Delta_S(A,B)$.

**Claim:** For some $f_1, \ldots, f_m$ function symbols in $L$,

$$PT \models \forall x \forall_{(i=1}^m (A(x, f_i(x)) \vee B(x, f_i(x))).$$

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Suppose not. Then \( \Gamma(y) = \{ \neg A(v,f(v)) \land \neg B(v,f(v)) : f \in L \} \) is consistent with PT. Let \( M \models PT \) with \( c \in M \) realizing \( \Gamma(y) \). Let \( M' \) be the substructure of \( M \) generated by \( c \). PT is essentially a universal theory (all existential sentences are witnessed in the standard part and hence by terms in the language), so \( M' \models PT \). But \( M' \models \forall y(A(c,y) \lor B(c,y)), \) a contradiction.

Thus, \( x \in S \) iff \( \forall m \ (A(x,f^*_1(x)) \lor B(x,f^*_1(x))) \), but this is polynomial time decidable. //

Similar arguments allow you to prove that for \( S \in NP \), \( PT \models "S \in co-NP" \) if \( P = NP \).

DeMillo and Lipton hoped that the model theory for PT would be sufficiently tractable to produce nonstandard models. Solovay destroyed these hopes by showing that PT has no recursive nonstandard models. This forced DeMillo and Lipton to conclude that "PT is almost as strong as PA". McAlon conjectured that models of PT are much harder to produce than models of PA. We confirm this below by showing that if \( M \) is a nonstandard model of PT then the atomic diagram of the reduct of \( M \) to the language of arithmetic has Turing degree above \( Q' \). We begin by showing that \( \text{Re}(M) \) is a Scott set. We do this by producing nonstandard initial segments of \( M \) which are models of PA.

Fix \( M \) a nonstandard model of PT. Let \( \log(x) \) be the polynomial time computable function \( \log(x) = \max\{y : 2^y \leq x\} \). We think of \( \log \) as a function from \( M \rightarrow M \) by taking the interpretation of the function symbol in \( L \) for \( \log \). Let \( c \in M \) be nonstandard. Let \( N = \{ m \in M : \text{ for every } n \in \omega \ \log(n)^c \geq m \} \), where \( \log(n)^c \) denotes
\[ \log(\log(\log(\ldots \log(x)))). \]
\[ \text{n times} \]

**Lemma 8.3**: (Solovay [D-L]). i) \( N \not\supseteq \omega \) and \( N \) is an initial segment of \( M \).

ii) \( N \) is closed under the function \( x \mapsto 2^x \).

iii) \( N \) is closed under all polynomial time computable functions.

Hence \( N \models PT \).

**Proof**: i) Clearly \( N \) is an initial segment of \( M \) and \( N \supseteq \omega \).

So we need only show \( N \) contains a nonstandard element. If \( x \in M \), let \( |x| \) denote the length of a binary representation of \( x \). Define a function

\[ \ell(n,x) = \begin{cases} 
  x & \text{if } n = 0 \\
  |\ell(n-1,x)| & \text{if } n > 0.
\end{cases} \]

Define \( \ell^*(n,x) = \ell(|n|,x) \), \( \ell^*(n,x) \) is polynomial time computable.

As \( c \notin \omega \), for every \( n \in \omega \), \( \ell^*(n,c) > 1 \). Define \( R(x,y) \iff \ell^*(x,y) = 1 \). It is easily seen that \( \forall y \ R(|y|,y) \). Thus \( \forall y \exists x \leq y \ R(x,y) \).

Define \( f_R(y) \) to be the least \( v < |y| \) so that \( R(v,y) \), \( f_R \) is polynomial time computable. Let \( \theta \) be the sentence \( \forall x \forall y (R(f_R(y),y) \land (x < f_R(y) \rightarrow \neg R(x,y))) \). As \( N \models \theta \), \( M \models \theta \). Let \( f_R(c) = d \). Note \( d > \omega \). We see that \( c \) is of the order

\[ 2 \quad \cdots \quad 2 \]
\[ \text{-d times.} \]

So \( \log(\ell(c)) \) is of the order

\[ 2 \quad \cdots \quad 2 \]
\[ \text{-d-n times.} \]
As \( d \) is nonstandard \( \lfloor d/2 \rfloor < d-n \) for \( n \in \omega \), where \( \lceil x \rceil \) is the greatest integer less than \( x \). Thus for each \( n \), \( \lfloor d/2 \rfloor < d-n < \log(n)c \). So \( \lfloor d/2 \rfloor \notin N \setminus \omega \).

ii) If \( a < \log(n+1)c \), \( 2^a < \log(n)c \). So if \( a \in N \), then \( 2^a \notin N \). (Note: 1) We are implicitly using the fact that \( \{(x,y): x = 2^y\} \) is polynomial time decidable. 2) In general, models of \( PT \) are not closed under exponentiation (the model \( M' \) used in the proof of 8.2 gives an example of one which is not).

iii) Let \( f \) be computable in time \( n^k \). Then for any \( x \), \( |f(x)| \leq |x|^k \). So \( f(x) \leq 2^{|x|^k} \leq x^{|x|^k} \). As \( x^{|x|^k} \) is dominated by \( 2^x \), for any nonstandard \( a \), \( f(a) \leq 2^a \). Thus by ii), if \( a \in N \), \( f(a) \in N \).

Lemma 8.4: \( N \models \Pi^*_0 \), where \( \Pi^*_0 \) denotes the theory with an induction schema for bounded formulas in the language \( L \).

Proof: We begin with an illustrative example which avoids the notation obscurities of the general case. Let \( P(x,y,z) \) be a polynomial time decidable predicate. Let \( \bar{a} \in N \) and suppose \( N \models \exists x \forall y < x P(x,y,\bar{a}) \). Let \( b \in N \) such that \( \forall y < b P(b,y,\bar{a}) \). We claim there is a least \( d \) with this property.

We define a new predicate \( R(w,z) \) by \( \forall y < |w| P(|w|,y,z) \).
\( R(w,z) \) is polynomial time decidable and \( N \models R(2^b,\bar{a}) \). Define \( Q(v,z) \) by \( \exists s < |v| R(s,z) \). \( Q \) is polynomial time decidable and \( N \models Q(2^b,\bar{a}) \). Define \( f(u,z) = \begin{cases} 0 & \text{if } Q(u,z) \\ \text{least } s \text{ } R(s,z) \text{ } Q(u,z). \end{cases} \)

\( f \) is polynomial time computable.
Let \( \theta \) be the sentence \( \forall u, z, x (\neg Q(u, z) \lor (Q(u, z) \land R(f(u, z), z)) \land (x < f(u, z) \rightarrow \gamma R(x, z))) \). Then \( N \models \theta \), so \( N \models \theta \). Let \( d = f(2^2, a) \).

Then \( d \) is least such that \( \forall y < |d| \ P(|d|, y, a) \). So \( |d| \) is the least \( x \) such that \( \forall y < x \ P(x, y, z) \).

We now extend these ideas to the general case. Suppose \( N \models \exists x \varphi(x, a) \), where \( \varphi(x, z) \) is \( L \) in the language \( L \). Let \( b \in N \) such that \( \varphi(b, a) \). We will show there is a least \( d \) such that \( \varphi(d, a) \).

We write \( \varphi(x, z) \) as \( Q_0 v_0 \leq t_0 \cdots Q_n v_n \leq t_n \), \( P(x, \bar{v}, \bar{z}_0, \bar{z}_1) \), where each \( t_i \) is either \( x \) or an element of \( \bar{z}_0, \bar{z}_0 \) and \( \bar{z}_1 \) are disjoint with union \( \bar{z} \) and \( P(x, \bar{v}, \bar{z}_0, \bar{z}_1) \) is polynomial time decidable. Break \( \bar{a} \) up into \( \bar{a}_0 \) and \( \bar{a}_1 \) in the obvious way.

Define \( R(y, w, z) \) by \( Q_0 v_0 \leq |s_0| \), \( Q_n v_n \leq |s_n| \), \( P(|y|, \bar{v}, |w|, \bar{z}_0, \bar{z}_1) \)
where, if \( t_i = x \), then \( s_i = y \) and if \( t_i = \bar{z}_0 \), then \( s_i = w_j \). \( R \) is polynomial time decidable and \( N \models R(\bar{z}_0, \bar{z}_0, \bar{a}_0) \), where \( \bar{a}_0 \) denotes \( <2^0, 2^1, \ldots, 2^n> \) where \( \bar{z}_0 \) is \( <i_0, \ldots, i_n> \). Define \( Q(v, \bar{v}, \bar{z}) \) by \( \exists s < |v| \ R(s, \bar{v}, \bar{z}) \). \( Q \) is polynomial time computable and \( N \models Q(2^2, 2^0, a_1) \). Define a Skolem function

\[
f(u, \bar{v}, \bar{z}) = \begin{cases} 
0 & \gamma Q(u, \bar{v}, \bar{z}) \\
\text{the least } s \text{ s.t. } R(s, \bar{v}, \bar{z}) \ Q(u, \bar{v}, \bar{a}) 
\end{cases}
\]

\( f \) is polynomial time computable. Let \( \theta \) be the sentence

\[
\forall u, \bar{z}_1, \bar{w}, x (\neg Q(u, \bar{w}, \bar{z}) \lor (Q(u, \bar{w}, \bar{z}) \land R(f(u, \bar{w}, \bar{z}), \bar{w}, \bar{z})) \\
\land (x < f(u, \bar{w}, \bar{z}) \rightarrow \gamma R(x, \bar{w}, \bar{z}_1))))
\]

\( N \models \theta \), so \( N \models \theta \). Let \( d = f(2^2, a_0, a_1) \). Then \( d \) is minimal such that \( R(d, a_0, a_1) \). So \( |d| \) is minimal such that \( \varphi(|d|, a) \). Thus \( N \models \exists \bar{v}_0 \).
In particular, $\bar{N}$ the reduct of $N$ to the language of arithmetic is a model of $I\Sigma_0$. This will allow us to use the following theorem of McAlloon.

**Theorem 8.5:** (McAlloon [Mc]). If $M \models I\Sigma_0$ is nonstandard, there is a nonstandard initial segment of $M$ modeling PA.

Fix an initial segment $I$ of $M$ such that $I \models PA$. We show $Re(M)$ is a Scott set by showing $Re(M) = Re(I)$.

**Lemma 8.6:** If $a \in M$ and $b \in M$ is nonstandard, there is $d < b$ such that $r(a) = r(d)$.

**Proof:** Let $F(n) = \max\{x: x! \leq n\}$. Let $\varphi(x,y)$ be the formula $x < y \Rightarrow \exists z < |x| \forall w < F(|x|) \forall v < w \ (v \text{ prime} \rightarrow (v|x| < v|z|))$. It is easily verified that $\varphi(x,y)$ is polynomial time computable. As $\mathbb{N} \models \forall x, y \varphi(x,y)$, so does $M$. In particular, $M \models \varphi(b,a)$. Thus, there is $d < |b|$ such that $\forall w < F(|b|) \forall v < w \ (v \text{ prime} \rightarrow (v|a| < v|d|))$.

As $F(|b|)$ is nonstandard, for $n \in \omega \ p_n | d\leftrightarrow p_n | a$. So $r(a) = r(d)$. //

**Corollary 8.7:** $Re(M) = Re(I)$, so $Re(M)$ is a Scott set.

**Corollary 8.8:** (Solovay). If $M \models PT$ is nonstandard, $\text{Diag}(M)$ $\nrightarrow 0$.

Here we use the fact that $r(a) \leq \text{Diag}(M)$ because the Euclidean algorithm works for sufficiently small numbers.

**Lemma 8.9:** $0' \in Re(M)$.

**Proof:** Let $f(x) = 0$ be a Diophantine equation. $M \models \exists x f(x) = 0$ iff $N \models \exists x f(x) = 0$. Further, if $N \models \exists x f(x) = 0$, then $I \models \exists x f(x) = 0$ and if $I \models \exists x f(x) = 0$, then $M \models \exists x f(x) = 0$, so $N \models \exists x f(x) = 0$. Hence
I and \( \mathbb{N} \) have the same Diophantine theory. By Martijasevic's theorem [D-M-R], \( \text{Th}_{\mathbb{N}}(I) = \text{Th}_{\mathbb{N}}(\mathbb{N}) \). By 1.17, \( \text{Th}_{\mathbb{N}}(I) \in \text{Re}(I) = \text{Re}(M) \). But \( \text{Th}_{\mathbb{N}}(\mathbb{N}) \equiv_T 0' \). //

Putting everything together we get the following theorem which proves McAloon's conjecture.

**Theorem 8.10:** If \( M \models PT \) is nonstandard and \( \bar{M} \) is the reduct of \( M \) to the language of arithmetic, then there is a \( \text{Diag}(M) \) enumeration of a Scott set containing \( 0' \).

We can prove a converse to 8.10.

**Lemma 8.11:** Let \( S \) be a Scott set containing \( 0' \). There is a recursively saturated \( M \models PT \) with \( \text{Re}(M) = S \).

**Proof:** Let \( T_0 = \{ \exists x \ P(x) : \mathbb{N} \models \exists x \ P(x) \} \cup \{ \forall x : P(x) : \mathbb{N} \models P(x) \} \).

Clearly, \( T_0 \leq_T 0' \). Suppose \( M \models T_0^* \) and \( PT \models \exists x \forall y \ P(x,y) \). Then there is \( n \in \omega \) such that \( \mathbb{N} \models \forall y \ P(n,y) \). Then \( \forall y \ P(n,y) \in T_0^* \). So \( M \models \exists x \forall y \ P(x,y) \). Hence \( M \models PT \).

Let \( C = \{ c_0, c_1, c_2, \ldots \} \) be a set of Henkin constants. Let \( T_1(\bar{v}), T_2(\bar{v}) \ldots \) list all partial types with finitely many constants from \( C \) which are coded in \( S \). Let \( \varphi_0, \varphi_1, \varphi_2 \ldots \) list all \( L(C) \) sentences. We build \( T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots \) so that each \( T_i \in S \) and each \( T_i \) has finitely many constants.

**Step 0:** \( T_0 = T_0 \).

**Step s = 4 \( \to \) s + 1:** Let \( T_s = T_{s-1} \cup \{ \varphi_n \} \) to maintain consistency.
Step $s = 4 + 2$: If we have added $\exists x \psi(x)$ at step $s - 1$ pick a new constant $c$ and let $T_s = T_{s-1} \cup \{\psi(c)\}$.

Step $s = 4 + 3$: If $\tau_n(\vec{v})$ is consistent with $T_{s-1}$, pick a new $c$ and let $T_s = T_{s-1} \cup \tau_n(c)$.

Step $s = 4 + 4$: Let $T_s$ be a completion of $T_{s-1}$ inside $S$, adding no new constants.

Let $M$ be the Henkin model of $\mathit{UT}_n$. Step $4 + 3$ insures $M$ is recursively saturated and $\mathit{Re}(M) \supseteq S$. Step $4 + 4$ insures $\mathit{Re}(M) \subseteq S$. //

**Theorem 8.12:** If $d$ enumerates a Scott set $S$ containing $0^1$, then there is a recursively saturated $M \models \mathit{PT}$ with $\mathit{Re}(M) = S$ and $\mathit{Diag}(M) \equiv_T d$.

**Proof:** Clear from 8.11 and 5.13. //
APPENDIX

Proposition A1: There is $M \models I\Sigma_0$ and a nonstandard $a \in M$ such that $[0,a]$ is not recursively saturated.

We define a theory $P_{\text{top}}$ in the language of arithmetic, with the following axioms.

- $\forall x \, s(x) \neq 0$
- $\exists! x \, s(x) = x$ (we denote this as $\alpha$)
- $\forall x \, \forall y ((x \neq \alpha \land y = s(y)) \rightarrow x = y)$
- $\forall x (x+0 = x)$
- $\forall x \, \forall y (x+y \cdot s(y) = s(x+y))$
- $\forall x (x \cdot 0 = 0)$
- $\forall x \, \forall y (x \cdot s(y) = x \cdot y + x)$
- $\exists x ((\varphi(0,x) \land \forall y (\varphi(y,x) \rightarrow \varphi(y+1,x)) \rightarrow \forall y \varphi(y,x)).$

$P_{\text{top}}$ is the natural theory of structures $[0,a]$ where $a \in M$ and $M$ is a model of a reasonably rich arithmetic theory, like $I\Sigma_0$. Paris showed that in fact, every model of $P_{\text{top}}$ arises this way.

Lemma A2: (Paris [G-M-W]). Every model $\alpha$ of $P_{\text{top}}$ admits an end extension $\mathcal{L}$ to a model of $I\Sigma_0$.

$\mathcal{L}$ is obtained in a natural way by considering polynomials over $\alpha$.

To prove A1 then, it is sufficient to show that $P_{\text{top}}$ has a nonrecursively saturated model. But this is easy. The induction axioms of $P_{\text{top}}$ guarantee that every model has definable Skolem functions. Thus, if $\alpha \models P_{\text{top}}$, we can take $\mathcal{L} \prec \alpha$ pointwise definable. $\mathcal{L}$ cannot be recursively saturated as we omit the type $\Gamma(v) = \{ \varphi(v) \rightarrow \exists w \neq n \varphi(w): \varphi \}$. Thus, by Paris's lemma, there is $M \models I\Sigma_0$ with $a \in M$ such that $\mathcal{L} \equiv [0,a]$. 65
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