Math 215: Introduction to Advanced Mathematics
Problem Set 11

Due: Friday November 30

1) a) Prove that the interval $(0,1)$ is equipotnet with the interval (a,b). [Note: the interval $(c,d)=\{x\in\mathbb{R}: c<x<d\}$.]

b) Prove that the interval $(0,1)$ is equipotent with the interval $(0,+\infty)$.

c) Prove that the interval $(0,+\infty)$ is equipotent with $\mathbb{R}$. Conclude that $(0,1)$ is equipotent with $\mathbb{R}$.

[HINT: For this problem you can use familiar functions from algebra and calculus.]

2) If A is any set we define $A^2 = A\times A$ and

$$A^n = \underbrace{A\times\ldots\times A}_{n-\text{times}}.$$

We also think of $A^n = \{(a_1,\ldots,a_n): n\in A\}$. Let $Seq(A)=\bigcup_{n\in\mathbb{N}} A^n$. Then $Seq(A)$ is the set of all finite sequences from A.

a) Prove that if A is countable, then A^n is countable for all n. [Hint: This should be an easy induction.]

b) Prove that if A is countable, then $Seq(A)$ is countable.

3) Suppose A and B are nonempty.

a) Prove that if there is a surjection $f: A\to B$, then there is an injection $g: B\to A$. [Hint: In an earlier homework we showed that there is $g: B\to A$ such that $f\circ g = Id_B$.]

b) Prove that if there is an injection $f: B\to A$, there is a surjection $g: A\to B$.

Taken together these show that we could equivalently define $|A|\le|B|$ by there is a injection from $A\to B$ or there is an surjection from $B\to A$.

(5pt bonus) If $f: A\times B\to C$. For each $a\in A$, we get a function $f_a\in\mathcal{F}(B,C)$, where $f_a: B\to C$ is given by $f_a(b)=f(a,b)$.

Let $\Phi:\mathcal{F}(A\times B,C)\to\mathcal{F}(A,\mathcal{F}(\mathcal{B},\mathcal{C}))$ be defined so that for $f: A\times B\to C$, $\Phi(f): A\to\mathcal{F}(B,C)$ is the function such that $\Phi(f)(a)=f_a$.

Prove that Φ is a bijection. This gives an argument that even for infinite sets

$$\left(|C|^{|A|}\right)^{|B|} = |C|^{|A||B|}.$$