Math 215: Introduction to Advanced Mathematics
Problem Set 11

Due: Monday December 1

1) a) Prove that the interval $(0,1)$ is equipotnet with the interval (a,b). [Note: the interval $(c,d) = \{x \in \mathbb{R} : c < x < d\}$.]

b) Prove that the interval $(0,1)$ is equipotent with the interval $(0,+\infty)$.

c) Prove that $\mathbb{R}$ is equipotent with the the interval $(0,+\infty)$. Conclude that $(0,1)$ is equipotent with $\mathbb{R}$.
[HINT: For this problem you can use familiar functions from algebra and calculus.]

2) If $A_1, A_2, \ldots$ are sets we let

$$A = \bigcup_{i=1}^{\infty} A_i = \{x : x \in A_i \text{ for some } i = 1, 2, \ldots\}.$$

Suppose each A_i is countable and $f_i : \mathbb{N} \to A_i$ is a surjection. Let $f : \mathbb{N}\times\mathbb{N} \to A$ be the function $f(i,j) = f_i(j)$.

a) Prove that f is a surjection.

b) Prove that A is countable. We have proved that *a countable union of countable sets is countable.*

3) If A is any set we define $A^2 = A \times A$ and

$$A^n = \underbrace{A \times \ldots \times A}_{n-\text{times}}.$$

We also think of $A^n = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in A\}$. Let $Seq(A) = \bigcup_{n=1}^{\infty} A^n$. Then $Seq(A)$ is the set of all finite sequences from A.

a) Prove that if A is countable, then A^n is countable for all n. [Hint: This should be an easy induction.]

b) Prove that if A is countable, then $Seq(A)$ is countable. [Hint: Use 2)]

(5pt bonus) If $f : A \times B \to C$. For each $a \in A$, we get a function $f_a \in \mathcal{F}(B,C)$, where $f_a : B \to C$ is given by $f_a(b) = f(a,b)$.

Let $\Phi : \mathcal{F}(A \times B, C) \to \mathcal{F}(A, \mathcal{F}(\mathcal{B}, \mathcal{C}))$ be defined so that for $f : A \times B \to C$, $\Phi(f) : A \to \mathcal{F}(B, C)$ is the function such that $\Phi(f)(a) = f_a$.

Prove that Φ is a bijection. This gives an argument that even for infinite sets

$$\left(|C|^{|A|}\right)^{|B|} = |C|^{|A||B|}.$$