Math 215: Introduction to Advanced Mathematics
Problem Set 10

Due Wednesday November 22

1) A standard deck of cards has 4 suits and each suit has 13 cards 2, 3, \ldots, 10, J, Q, K, A. In draw poker your are dealt 5 cards.
   a) How many 5 card poker hands are there? [Note: The order you are dealt
   the cards doesn’t matter. If you are dealt the A♥, 5♦, A♠, K♠, and then
   the 3♣, you have the same hand as if you were dealt 5♦, K♠, 3♣, A♥, and
   then the A♠.]
   b) A flush is when all 5 cards are from the same suit. How many ways are
   there to be dealt a flush.
   To calculate the probability of being dealt a flush, divide your answer to b)
   by your answer to a).

2) Let \( X \) be a finite set with \( |X| = n \) and let \( 0 \leq r \leq n \).
   Let \( F : \mathcal{P}_r(X) \to \mathcal{P}_{n-r}(X) \) be the function
   \( F(A) = X - A \). Prove that \( X \) is a bijection and conclude that
   \[
   \binom{n}{r} = \binom{n}{n-r}
   \]

3) a) Suppose \( X \) and \( Y \) are disjoint sets. Let
   \[
   A = \bigcup_{i=0}^{k} \mathcal{P}_i(x) \times \mathcal{P}_{k-i}(Y) = (\mathcal{P}_0(X) \times \mathcal{P}_k(Y)) \cup (\mathcal{P}_1(X) \times \mathcal{P}_{k-1}(Y)) \cup \ldots \cup (\mathcal{P}_k(X) \times \mathcal{P}_0(Y)).
   \]
   Let \( F : A \to \mathcal{P}_k(X \cup Y) \) be the function
   \[
   F(A \times B) = A \cup B.
   \]
   Prove that \( F \) is a bijection.
   b) Use a) to conclude that
   \[
   \binom{m+n}{k} = \sum_{i=0}^{r} \binom{m}{i} \binom{n}{r-i}.
   \]

4) (5pt Bonus) Let \( X, Y, Z \) be nonempty sets. Suppose \( f : X \times Y \to Z \) and
   \( y \in Y \), let \( f_y : X \to Z \) be the function
   \[
   f_y(x) = f(x, y).
   \]
   Define \( \Phi : \mathcal{F}(X \times Y, Z) \to \mathcal{F}(Y, \mathcal{F}(X, Z)) \) as follows: For \( f : X \times Y \to Z \), let
   \( \Phi(f) : Y \to \mathcal{F}(X, Z) \) be the function \( y \mapsto f_y \). Prove that \( \Phi \) is a bijection.
   For finite sets \( X, Y, Z \), this shows
   \[
   |Z|^{\ |X| \cdot |Y|} = \left( |Z|^{|X|} \right)^{|Y|}.
   \]