Groups of order 6

Suppose G is a group of order 6. We will prove that $G \cong \mathbb{Z}_6$ or $G \cong D_3$.
case 1: G has an element a of order 6.
Then $G = \langle a \rangle$ is cyclic and $G \cong \mathbb{Z}_6$.
case 2: G has no elements of order 6.
We have argued in class that:
- G has an element a of order 3.
- If $c \in G$ and $c \notin \langle a \rangle$, then c has order 2.

Choose $b \in G$ with $b \notin \langle a \rangle$. Since $G = \langle a \rangle \cup b\langle a \rangle$, the elements of G are $\{e, a, a^2, b, ba, ba^2\}$. Then

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba^2.$$

We claim that this is enough information to completely fill in the group table for G. For example:

$a(ba) = (ab)a = (ba^2)a = ba^3 = b$
$a(ba^2) = (ab)a^2 = (ba^2)a^2 = ba$
$a^2b = a(ab) = a(ba^2) = (ab)a^2 = (ba^2)a^2 = ba$
$a^2(ba) = a(ab)a = a(ba^2)a = ab = ba^2$
$a^2(ba^2) = a(ab)a^2 = a(ba^2)a^2 = (ab)a = (ba^2)a = b$
$(ba)b = b(ab) = b(ba^2) = a^2$
$(ba)(ba) = b(ab)a = b(ba^2)a = e$
$(ba)(ba^2) = b(ab)a^2 = b(ba^2)a^2 = a$
$(ba^2)b = ba(ab) = ba(ba^2) = b(ab)a^2 = b(ba^2)a^2 = a$
$(ba^2)(ba) = (ba)(ab)a = (ba)(ba^2)a = b(ab) = b(ba^2) = a^2$
$(ba^2)(ba^2) = (ba)(ab)a^2 = (ba)(ba^2)a^2 = b(ab)a = b(ba^2)a = e$

	e	a	a^2	b	ba	ba^2
e	e	a	a^2	b	ba	ba^2
a	a	a^2	e	ba^2	b	ba
a^2	a^2	e	a	ba	ba^2	b
b	b	ba	ba^2	e	a	a^2
ba	ba	ba^2	b	a^2	e	a
ba^2	ba^2	b	ba	a	a^2	e

Compare this to the group table for D_3 and notice that if $\phi(e) = R_0$, $\phi(a) = R_{120}$, $\phi(a^2) = R_{240}$, $\phi(b) = L_1$, $\phi(ba) = L_2$ and $\phi(ba^2) = L_3$, then $\phi : G \to D_3$ is an isomorphism.

$\circ$	R_0	R_{120}	R_{240}	L_1	L_2	L_3
R_0	R_0	R_{120}	R_{240}	L_1	L_2	L_3
R_{120}	R_{120}	R_{240}	R_0	L_3	L_1	L_2
R_{240}	R_{240}	R_0	R_{120}	L_2	L_3	L_1
L_1	L_1	L_2	L_3	R_0	R_{120}	R_{240}
L_2	L_2	L_3	L_1	R_{240}	R_0	R_{120}
L_3	L_3	L_1	L_2	R_{120}	R_{240}	R_0