Math 330: Abstract Algebra
Final Exam Study Guide

The Final exam will be on Friday December 13 1:00-3:00.
The exam will be cumulative, though it will focus a bit more on material
from the second half of the course.

A detailed list of material from the text that we have covered in the course
is on the web at:
http://www.math.uic.edu/~marker/math330/topics.html

Exam Week Office Hours
Wednesday 3:00–6:00
Thursday 10:00–12:00, 3:00–6:00
Friday 9:00–11:00

A large portion of the exam test that you know the important definitions
and theorems, and that you can apply them to study specific examples.
The exam will also have one or two proofs, at the level of moderate homework
problems.

Here is a list of some of the things you should know for the exam.

1. Know all important definitions and how to apply them: groups, subgroups,
cyclic groups, order of a group, order of an element, Abelian groups, permutation
groups, isomorphism, automorphism, cosets, direct product, normal subgroup,
factor group, group homomorphisms, kernel, rings, units of rings, zero divisors,
integral domains, fields, ideals, factor rings, prime ideals, maximal ideals, ring
homomorphism, field of quotients, principal ideal domains, irreducible elements in
a ring, unique factorization domains, extension fields, splitting fields, algebraic
and transcendental elements ....

2. Know how to state and use the important theorems and their corollaries:
Fundamental Theorem of Cyclic Groups, Caley’s Theorem, Lagrange’s Theo-
rem, Cauchy’s Theorem for Abelian Groups, First Isomorphism Theorem for
Groups, Fundamental Theorem for Finite Abelian Groups, existence of factor
rings, the characterization of prime and maximal ideals in terms of factor rings,
the division algorithm for polynomial rings, the polynomial ring $F[X]$ is prin-
ciple ideal domain and a unique factorization domain, Fundamental Theorem of
Field Theory, characterization of finite fields...

3. Understand the important examples of groups and how to work with them,
such as: $\mathbb{Z}, \mathbb{R}, \mathbb{R}^*, \mathbb{C}, \mathbb{C}^*, \mathbb{Z}_n, U(n), D_n, S_n, A_n, GL(n, \mathbb{R}), SL(n, \mathbb{R}), GL(n, \mathbb{Z}_p), \ldots$

Understand the important examples of rings and how to work with them for example: $\mathbb{Z}_n, F[X]$ for $F$ a field, $F[X]/\langle p \rangle$ for $p \in F[X]$ irreducible.
1) Define the following concepts:
   a) \( H \) is a normal subgroup of \( G \) and \( K \) is the factor group \( G/H \).
   b) \( I \subset R \) is a maximal ideal;
   c) \( a \in R \) is irreducible.
   d) \( \phi : R \rightarrow S \) is a ring homomorphism.

2) Decide if the following statements are TRUE or FALSE. If FALSE give a counterexample or an explanation why the statement is false.
   a) If \( R \) is a commutative ring and \( I \subset R \) is an ideal, then \( R/I \) is commutative.
   b) If \( G \) is a finite Abelian group and \( n \) divides \( |G| \), then \( G \) has an element of order \( n \).
   c) If \( F \) is a field, then \( F[X, Y] \) is a principal ideal domain.
   d) There is a field \( F \) with \( |F| = 81 \).
   e) If \( G \) is a group, \( H \) is a subgroup of \( G \) and \( H \) is Abelian, then \( H \) is a normal subgroup of \( G \).
   f) If \( R \) is a ring and \( ab = ac \), then \( b = c \).
   g) If \( F \) is a field, \( I \subset F \) is an ideal, \( a \in F \setminus \{0\} \) and \( a \in I \), then \( I = F \).

3) State the following theorem:
   a) LaGrange’s Theorem
   b) The Fundamental Theorem of Field Theory

4) a) Find all Abelian groups (up to isomorphism) of order 16.
   b) Which of these groups is isomorphic to \( \mathbb{Z}_4 \times \mathbb{Z}_{24}/\langle(2, 4)\rangle \)?

5) a) Show that \( p(X) = X^3 + X + 1 \) is irreducible in \( \mathbb{Z}_2[X] \).
   b) Suppose \( F \) is an extension field of \( \mathbb{Z}_2 \), \( \alpha \in F \) and \( p(\alpha) = 0 \). Find \( a, b, c \in \mathbb{Z}_2 \) such that
      \[ a\alpha^2 + b\alpha + c = (\alpha^2 + 1)(\alpha + 1). \]

6) Find a noncyclic subgroup of order 4 in \( \mathbb{Z}_4 \oplus \mathbb{Z}_{10} \).

7) Show that the homomorphic image of a principal ideal domain is a principal ideal domain.

8) Suppose \( R \) is a commutative ring and \( I \subset R \) an ideal. Let
    \[ \sqrt{I} = \{ a \in R : a^n \in I \} \]
    for some \( n \).
    a) Show that \( \sqrt{I} \) is an ideal.
    b) Show that the factor ring \( R/\sqrt{I} \) has no nonzero nilpotent elements (recall that \( a \in R \) is nilpotent if \( a^n = 0 \) for some \( n = 1, 2, \ldots \)).