A group where the squares do not form a subgroup
Math 330: Abstract Algebra

We proved in class that if G is an Abelian group and $H = \{g^2 : g \in G\}$, then H is a subgroup. We also noticed that if G is the dihedral group D_n, then the squares form a subgroup (indeed they are a subgroup of the group of rotations).

We want to give an example of a nonAbelian group where the squares do not form a subgroup.

Let $G = SL(2, \mathbb{Z}_3)$, the group of 2×2 matricies with entries from $\mathbb{Z}_3$ and determinant 1 mod 3. The number of 2×2 matrices with entries from $\mathbb{Z}_3$ is $3^4 = 81$. Of those 33 have determinant 0, 24 have determinant 1, and 24 have determinant 2.

Calcuating we find that the following 10 matricies are squares in $SL(2, \mathbb{Z}_3)$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$.

Note that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \pmod 3$$

is not a square.

A second example is the group A_4. The group A_4 has 12 elements. See the group table on page 104. Note that $\alpha_6^2 = \alpha_{11}$, $\alpha_9^2 = \alpha_5$ and $\alpha_{11}\alpha_5 = \alpha_3$, but α_3 is not a square.