A group where the squares do not form a subgroup
Math 330: Abstract Algebra

We proved in class that if $G$ is an Abelian group and $H = \{g^2 : g \in G\}$, then $H$ is a subgroup. We also noticed that if $G$ is the dihedral group $D_n$, then the squares form a subgroup (indeed they are a subgroup of the group of rotations).

We want to give an example of a nonAbelian group where the squares do not form a subgroup.

Let $G = SL(2, \mathbb{Z}_3)$, the group of $2 \times 2$ matrices with entries from $\mathbb{Z}_3$ and determinant $1$ mod $3$. The number of $2 \times 2$ matrices with entries from $\mathbb{Z}_3$ is $3^4 = 81$. Of those 33 have determinant $0$, 24 have determinant $1$, and 24 have determinant $2$.

Calculating we find that the following 10 matrices are squares in $SL(2, \mathbb{Z}_3)$,

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}, \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
2 & 2 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 2 \\
1 & 2
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
2 & 2
\end{bmatrix}.
$$

Note that

$$
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}^2 = \begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 2 \\
2 & 1
\end{bmatrix} \pmod{3}
$$

is not a square.

A second example is the group $A_4$. The group $A_4$ has 12 elements. See the group table on page 104. Note that $\alpha_0^2 = \alpha_{11}$, $\alpha_5^2 = \alpha_5$ and $\alpha_{11}\alpha_5 = \alpha_3$, but $\alpha_3$ is not a square.