Math 413–Analysis I

Final Exam–Solutions

1)(15pt) Define the following concepts:

a) $(x_n)_{n=1}^{\infty}$ converges to L;

For all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n \geq N$.

b) $A \subseteq \mathbb{R}$ is compact;

If $(x_n)_{n=1}^{\infty}$ is a sequence of elements of A, there is a subsequence converging to an element of A.

c) $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c.

$\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$ for some $L \in \mathbb{R}$.

2) (10pt) State the following Theorems:

a) Intermediate Value Theorem;

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < c < f(b)$. Then there is $a < x < b$ with $f(x) = c$.

b) Nested Interval Property;

If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ where each I_i is a closed interval $[a_i, b_i]$, then there is $x \in \bigcap_{n=1}^{\infty} I_n$.

3) (15pt) State and prove the Monontone Convergence Theorem.

If $(x_n)_{n=1}^{\infty}$ is a bounded sequence and $x_1 \leq x_2 \leq \ldots$, then (x_n) converges.

Let L be the least upper bound of $\{x_n : n = 1, 2, \ldots\}$. Let $\epsilon > 0$. Since $L - \epsilon$ is not an upperbound, there is $N \in \mathbb{N}$ such that $L - \epsilon < x_n \leq L$ for all $n \geq N$.

4)(30pt) Decide if the following statements are TRUE or FALSE. If FALSE, give an example showing the statement is FALSE.

a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subseteq \mathbb{R}$ is bounded, then $f(A)$ is bounded.

TRUE. Find an M such that $A \subseteq [-M, M]$. Then $f(A) \subseteq f([-M, M])$ and the later set is compact and hence bounded.

b) Suppose $f : A \to \mathbb{R}$ is continuous and $(a_n)_{n=1}^{\infty}$ is a convergent sequence in A with $\lim a_n \in A$, then $(f(a_n))_{n=1}^{\infty}$ converges.

TRUE. (On the other hand, if we don't know that the limit is in A, then we can not conclude that the sequence of images is convergent.)

c) If $(x_n y_n)_{n=1}^{\infty}$, $(x_n)_{n=1}^{\infty}$ are convergent where $x_n > 0$ for all n, then $(y_n)_{n=1}^{\infty}$ is convergent.

FALSE. Consider $(y_n) = (1, 2, 3, 4, \ldots)$ and $(x_n) = (1, 1/4, \ldots, 1/n^2, \ldots)$, then $(x_n y_n) = (1, 1/2, 1/3, \ldots)$ converges even though (y_n) does not.

d) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a, then f is continuous at a.

TRUE.

e) If A is bounded and $f : A \to \mathbb{R}$ is continuous, then f is uniformly continuous.

FALSE. Let $f : (0, 1) \to \mathbb{R}$ be $f(x) = \frac{1}{x}$.

f) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable at a, then $g \circ f$ is differentiable at a.

FALSE. Let $f(x) = x - 1$ and $g(x) = |x|$. Then f and g are both differentiable at 1. But $g \circ f(x) = |x - 1!$ is not differentiable at 1.

5)(10pt) Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous at c and $g(c) \neq 0$. Prove that there is an open interval (a, b) such that $a < c < b$ and $g(x) \neq 0$ for all $x \in (a, b)$.

We have $|g(c)| > 0$. Let $\epsilon = |g(c)|/2$. Since g is continuous at c there is $\delta > 0$ such that $|g(x) - g(c)| < \epsilon$ if $|x - c| < \delta$. Thus if $|x - c| < \delta$, then $|g(x) - g(c)| < |g(c)|/2$. But

$$|g(c)| \leq |g(x)| + |g(x) - g(c)| < |g(x)| + |g(c)|/2.$$

Thus

$$0 < |g(c)|/2 < |g(x)|$$

for $x \in (c - \delta, c + \delta)$.

6) (10pt) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Suppose there are four distinct points w, x, y, z such that $f(w) = f(x)$, $f(y) = y$ and $f(z) = z$. Prove that there is a point u where $f'(u) = \frac{1}{2}$.

Since $f(w) = f(x)$, by Rolle's Theorem, there is c between w and x with $f'(c) = 0$. By the Mean Value Theorem, there is d betwee y and z with

$$f'(d) = \frac{f(y) - f(z)}{y - z} = \frac{y - z}{y - z} = 1.$$

By Darboux's Theorem, there is u between c and d with $f'(u) = \frac{1}{2}$.

7) (10pt) Suppose $f : \mathbb{R} \to \mathbb{R}$. Suppose $\lim\limits_{x\to c} f(x) \neq L$. Prove that there is a sequence $(x_n)_{n=1}^{\infty}$ converging to c such that $(f(x_n))_{n=1}^{\infty}$ does not converge to L.

Since $\lim\limits_{x\to c} f(x) \neq L$, there is $\epsilon > 0$ such that for all $\delta > 0$ there is x such that $0 < |x - c| < \delta$ and $|f(x) - L| > \epsilon$.

For each $n \in \mathbb{N}$ choose x_n such that $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| > \epsilon$. Then $(x_n) \to c$, but $(f(x_n)) \not\to L$.