1)(15pt) Define the following concepts:
   a) \((x_n)_{n=1}^\infty\) converges to \(L\);
      
      For all \(\epsilon > 0\) there is an \(N \in \mathbb{N}\) such that \(|x_n - L| < \epsilon\) for all \(n \geq N\).
   b) \(A \subseteq \mathbb{R}\) is compact;
      
      If \((x_n)_{n=1}^\infty\) is a sequence of elements of \(A\), there is a subsequence converging
to an element of \(A\).
   c) \(f : \mathbb{R} \to \mathbb{R}\) is differentiable at \(c\).
      
      \[
      \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L \text{ for some } L \in \mathbb{R}.
      \]

2) (10pt) State the following Theorems:
   a) Intermediate Value Theorem;
      
      Suppose \(f : [a, b] \to \mathbb{R}\) is continuous and \(f(a) < c < f(b)\). Then there is
      \(a < x < b\) with \(f(x) = c\).
   b) Nested Interval Property;
      
      If \(I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots\) where each \(I_i\) is a closed interval \([a_i, b_i]\), then there
      is \(x \in \bigcap_{n=1}^\infty I_n\).

3) (15pt) State and prove the Monontone Convergence Theorem.
   If \((x_n)_{n=1}^\infty\) is a bounded sequence and \(x_1 \leq x_2 \leq \ldots\), then \((x_n)\) converges.
      
      Let \(L\) be the least upper bound of \(\{x_n : n = 1, 2, \ldots\}\). Let \(\epsilon > 0\). Since
      \(L - \epsilon\) is not an upperbound, there is \(N \in \mathbb{N}\) such that \(L - \epsilon < x_n \leq L\) for all
      \(n \geq N\).

4)(30pt) Decide if the following statements are TRUE or FALSE. If FALSE, give an example showing the statement is FALSE.
   a) Suppose \(f : \mathbb{R} \to \mathbb{R}\) is continuous and \(A \subseteq \mathbb{R}\) is bounded, then \(f(A)\) is
      bounded.
      
      TRUE. Find an \(M\) such that \(A \subseteq [-M, M]\). Then \(f(A) \subseteq f([-M, M])\)
      and the later set is compact and hence bounded.
   b) Suppose \(f : A \to \mathbb{R}\) is continuous and \((a_n)_{n=1}^\infty\) is a convergent sequence in
      \(A\) with \(\lim a_n \in A\), then \((f(a_n))_{n=1}^\infty\) converges.
 TRUE. (On the other hand, if we don’t know that the limit is in $A$, then we can not conclude that the sequence of images is convergent.)

c) If $(x_n y_n)_{n=1}^\infty$, $(x_n)_{n=1}^\infty$ are convergent where $x_n > 0$ for all $n$, then $(y_n)_{n=1}^\infty$ is convergent.

FALSE. Consider $(y_n) = (1, 2, 3, 4, \ldots)$ and $(x_n) = (1, 1/4, \ldots, 1/n^2, \ldots)$, then $(x_n y_n) = (1, 1/2, 1/3, \ldots)$ converges even though $(y_n)$ does not.

d) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a$, then $f$ is continuous at $a$.

TRUE.

e) If $A$ is bounded and $f : A \to \mathbb{R}$ is continuous, then $f$ is uniformly continuous.

FALSE. Let $f : (0, 1) \to \mathbb{R}$ be $f(x) = \frac{1}{x}$.

f) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable at $a$, then $g \circ f$ is differentiable at $a$.

FALSE. Let $f(x) = x - 1$ and $g(x) = |x|$. Then $f$ and $g$ are both differentiable at 1. But $g \circ f(x) = |x - 1|$ is not differentiable at 1.

5) (10pt) Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous at $c$ and $g(c) \neq 0$. Prove that there is an open interval $(a, b)$ such that $a < c < b$ and $g(x) \neq 0$ for all $x \in (a, b)$.

We have $|g(c)| > 0$. Let $\epsilon = |g(c)|/2$. Since $g$ is continuous at $c$ there is $\delta > 0$ such that $|g(x) - g(c)| < \epsilon$ if $|x - c| < \delta$. Thus if $|x - c| < \delta$, then $|g(x) - g(c)| < |g(c)|/2$. But

$$|g(c)| \leq |g(x)| + |g(x) - g(c)| < |g(x)| + |g(c)|/2.$$ 

Thus

$$0 < |g(c)|/2 < |g(x)|$$

for $x \in (c - \delta, c + \delta)$.

6) (10pt) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Suppose there are four distinct points $w, x, y, z$ such that $f(w) = f(x), f(y) = y$ and $f(z) = z$. Prove that there is a point $u$ where $f'(u) = \frac{1}{2}$.

Since $f(w) = f(x)$, by Rolle’s Theorem, there is $c$ between $w$ and $x$ with $f'(c) = 0$. By the Mean Value Theorem, there is $d$ between $y$ and $z$ with

$$f'(d) = \frac{f(y) - f(z)}{y - z} = \frac{y - z}{y - z} = 1.$$
By Darboux’s Theorem, there is $u$ between $c$ and $d$ with $f'(u) = \frac{1}{2}$.

7) (10pt) Suppose $f : \mathbb{R} \to \mathbb{R}$. Suppose $\lim_{x \to c} f(x) \neq L$. Prove that there is a sequence $(x_n)_{n=1}^{\infty}$ converging to $c$ such that $(f(x_n))_{n=1}^{\infty}$ does not converge to $L$.

Since $\lim_{x \to c} f(x) \neq L$, there is $\epsilon > 0$ such that for all $\delta > 0$ there is $x$ such that $0 < |x - c| < \delta$ and $|f(x) - L| > \epsilon$.

For each $n \in \mathbb{N}$ choose $x_n$ such that $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| > \epsilon$. Then $(x_n) \to c$, but $(f(x_n)) \not\to L$. 