Math 413 Analysis I
Midterm 1
October 17, 2003

1)(20pt) Give complete definitions of the following concepts:

a) $\lim_{x \to a} f(x) = L$ where $f : A \to \mathbb{R}$

a is a limit point of A and for all $\epsilon > 0$ there is $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

b) $(a_n)_{n=1}^{\infty}$ converges to L;

For all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$.

c) $O \subseteq \mathbb{R}$ is open;

If $x \in O$ there is $\epsilon > 0$ such that $V_\epsilon(x) \subseteq O$.

2) (20pt) Give complete statements of the following results

a) Nested Interval Property

If $I_1 \supseteq I_2 \supseteq \ldots$ are nonempty closed intervals, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

b) Bolzano–Weierstrass Theorem.

Every bounded sequence has a convergent subsequence.

3)(30pt) Decide if the following statements are TRUE or FALSE. If FALSE, give an example demonstrating it is FALSE.

a) If $A \subseteq \mathbb{R}$ is bounded above and nonempty there is $\alpha \in A$ a least upper bound for A.

FALSE. Let $A = [0, 1)$. The the least upper bound is $1 \notin A$.

b) If $(a_n)_{n=1}^{\infty}$ is an unbounded monotonic sequence, then every subsequence is divergent.

TRUE.

c) If $O_1, O_2, \ldots$ are open, then $\displaystyle\bigcup_{n=1}^{\infty} O_n$ is open.

TRUE.

d) Every open cover of $(-1, 1)$ has a finite subcover.

FALSE. Let $O_n = (-1 + \frac{1}{n}, 1)$ for $n = 1, 2, \ldots$. Then $O_1, O_2, O_3, \ldots$ is an open cover of $(-1, 1)$ with no finite subcover.

e) If $(a_n)_{n=1}^{\infty}$ is bounded and $(a_n + b_n)_{n=1}^{\infty}$ is Cauchy, then $(b_n)_{n=1}^{\infty}$ is Cauchy.

FALSE. Let (a_n) be the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ and (b_n) be the sequence $(1, 0, 1, 0, \ldots)$. Then $(a_n + b_n)$ is the sequence $(1, 1, 1, \ldots)$.

f) If O is open and $\mathbb{Q} \subseteq O$, then $O = \mathbb{R}$.

FALSE. Let $O = (-\infty, \sqrt{2}) \cup (\sqrt{2}, +\infty)$.

4)(15pt) Suppose the sequence $(x_n)_{n=1}^{\infty}$ converges to 0 and the sequence $(y_n)_{n=1}^{\infty}$ is bounded. Prove that that $\lim x_n y_n = 0$.

Let $\epsilon > 0$. Suppose $|y_n| \leq M$ for all n. We want $|x_n y_n| < \epsilon$. It would suffice to have $M|x_n| < \epsilon$ or $|x_n| < \epsilon/M$. Since $\lim x_n = 0$, there is N such that $|x_n| < \epsilon/M$ for all $n \geq M$. Then $|x_n y_n| < \epsilon$ for all $n \geq M$ and $\lim x_n y_n = 0$.

5)(10pt) Suppose $A \subset [0, 1]$ is infinite. Prove that there is $\alpha \in [0, 1]$ a limit point of A.

Let $a_1, a_2, a_3, \ldots$ be distinct elements of A. This is a bounded sequence. By the Bolzano–Weierstrass Theorem, $(a_n)_{n=1}^{\infty}$ has a convergent subseqeuence. That sequence converges to some α a limit point of A. Since $[0, 1]$ is closed $\alpha \in [0, 1]$.