1)(20pt) Give complete definitions of the following concepts:

a) \( \lim_{x \to a} f(x) = L \) where \( f : A \to \mathbb{R} \)

\( a \) is a limit point of \( A \) and for all \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( x \in A \) and \( 0 < |x - a| < \delta \), then \( |f(x) - L| < \epsilon \).

b) \((a_n)_{n=1}^{\infty}\) converges to \( L \);

For all \( \epsilon > 0 \) there is \( N \in \mathbb{N} \) such that \( |a_n - L| < \epsilon \) for all \( n \geq N \).

c) \( O \subseteq \mathbb{R} \) is open;

If \( x \in O \) there is \( \epsilon > 0 \) such that \( V_\epsilon(x) \subseteq O \).

2) (20pt) Give complete statements of the following results

a) Nested Interval Property

If \( I_1 \supseteq I_2 \supseteq \ldots \) are nonempty closed intervals, then \( \bigcap_{n=1}^{\infty} I_n \) is nonempty.

b) Bolzano–Weierstrass Theorem.

Every bounded sequence has a convergent subsequence.

3)(30pt) Decide if the following statements are TRUE or FALSE. If FALSE, give an example demonstrating it is FALSE.

a) If \( A \subseteq \mathbb{R} \) is bounded above and nonempty there is \( \alpha \in A \) a least upper bound for \( A \).

FALSE. Let \( A = [0, 1) \). The the least upper bound is \( 1 \notin A \).

b) If \((a_n)_{n=1}^{\infty}\) is an unbounded monotonic sequence, then every subsequence is divergent.

TRUE.

c) If \( O_1, O_2, \ldots \) are open, then \( \bigcup_{n=1}^{\infty} O_n \) is open.

TRUE.

d) Every open cover of \((-1, 1)\) has a finite subcover.

FALSE. Let \( O_n = (-1 + \frac{1}{n}, 1) \) for \( n = 1, 2, \ldots \). Then \( O_1, O_2, O_3, \ldots \) is an open cover of \((-1, 1)\) with no finite subcover.
e) If \((a_n)_{n=1}^\infty\) is bounded and \((a_n + b_n)_{n=1}^\infty\) is Cauchy, then \((b_n)_{n=1}^\infty\) is Cauchy.

FALSE. Let \((a_n)\) be the sequence \((0, 1, 0, 1, 0, 1, \ldots)\) and \((b_n)\) be the sequence \((1, 0, 1, 0, \ldots)\). Then \((a_n + b_n)\) is the sequence \((1, 1, 1, \ldots)\).

f) If \(O\) is open and \(\mathbb{Q} \subseteq O\), then \(O = \mathbb{R}\).

FALSE. Let \(O = (-\infty, \sqrt{2}) \cup (\sqrt{2}, +\infty)\).

4)(15pt) Suppose the sequence \((x_n)_{n=1}^\infty\) converges to 0 and the sequence \((y_n)_{n=1}^\infty\) is bounded. Prove that that \(\lim x_n y_n = 0\).

Let \(\epsilon > 0\). Suppose \(|y_n| \leq M\) for all \(n\). We want \(|x_n y_n| < \epsilon\). It would suffice to have \(M|x_n| < \epsilon\) or \(|x_n| < \epsilon/M\). Since \(\lim x_n = 0\), there is \(N\) such that \(|x_n| < \epsilon/M\) for all \(n \geq M\). Then \(|x_n y_n| < \epsilon\) for all \(n \geq M\) and \(\lim x_n y_n = 0\).

5)(10pt) Suppose \(A \subset [0, 1]\) is infinite. Prove that there is \(\alpha \in [0, 1]\) a limit point of \(A\).

Let \(a_1, a_2, a_3, \ldots\) be distinct elements of \(A\). This is a bounded sequence. By the Bolzano–Weierstrass Theorem, \((a_n)_{n=1}^\infty\) has a convergent subsequence. That sequence converges to some \(\alpha\) a limit point of \(A\). Since \([0, 1]\) is closed \(\alpha \in [0, 1]\).