

**Math 413 Analysis I**  
**Midterm 1**  
**October 17, 2003**

1)(20pt) Give complete definitions of the following concepts:

a)  $\lim_{x \rightarrow a} f(x) = L$  where  $f : A \rightarrow \mathbb{R}$

$a$  is a limit point of  $A$  and for all  $\epsilon > 0$  there is  $\delta > 0$  such that if  $x \in A$  and  $0 < |x - a| < \delta$ , then  $|f(x) - L| < \epsilon$ .

b)  $(a_n)_{n=1}^{\infty}$  converges to  $L$ ;

For all  $\epsilon > 0$  there is  $N \in \mathbb{N}$  such that  $|a_n - L| < \epsilon$  for all  $n \geq N$ .

c)  $O \subseteq \mathbb{R}$  is open;

If  $x \in O$  there is  $\epsilon > 0$  such that  $V_{\epsilon}(x) \subseteq O$ .

2) (20pt) Give complete statements of the following results

a) Nested Interval Property

If  $I_1 \supseteq I_2 \supseteq \dots$  are nonempty closed intervals, then  $\bigcap_{n=1}^{\infty} I_n$  is nonempty.

b) Bolzano–Weierstrass Theorem.

Every bounded sequence has a convergent subsequence.

3)(30pt) Decide if the following statements are TRUE or FALSE. If FALSE, give an example demonstrating it is FALSE.

a) If  $A \subseteq \mathbb{R}$  is bounded above and nonempty there is  $\alpha \in A$  a least upper bound for  $A$ .

FALSE. Let  $A = [0, 1)$ . The the least upper bound is  $1 \notin A$ .

b) If  $(a_n)_{n=1}^{\infty}$  is an unbounded monotonic sequence, then every subsequence is divergent.

TRUE.

c) If  $O_1, O_2, \dots$  are open, then  $\bigcup_{n=1}^{\infty} O_n$  is open.

TRUE.

d) Every open cover of  $(-1, 1)$  has a finite subcover.

FALSE. Let  $O_n = (-1 + \frac{1}{n}, 1)$  for  $n = 1, 2, \dots$ . Then  $O_1, O_2, O_3, \dots$  is an open cover of  $(-1, 1)$  with no finite subcover.

e) If  $(a_n)_{n=1}^{\infty}$  is bounded and  $(a_n + b_n)_{n=1}^{\infty}$  is Cauchy, then  $(b_n)_{n=1}^{\infty}$  is Cauchy.

FALSE. Let  $(a_n)$  be the sequence  $(0, 1, 0, 1, 0, 1, \dots)$  and  $(b_n)$  be the sequence  $(1, 0, 1, 0, \dots)$ . Then  $(a_n + b_n)$  is the sequence  $(1, 1, 1, \dots)$ .

f) If  $O$  is open and  $\mathbb{Q} \subseteq O$ , then  $O = \mathbb{R}$ .

FALSE. Let  $O = (-\infty, \sqrt{2}) \cup (\sqrt{2}, +\infty)$ .

4)(15pt) Suppose the sequence  $(x_n)_{n=1}^{\infty}$  converges to 0 and the sequence  $(y_n)_{n=1}^{\infty}$  is bounded. Prove that  $\lim x_n y_n = 0$ .

Let  $\epsilon > 0$ . Suppose  $|y_n| \leq M$  for all  $n$ . We want  $|x_n y_n| < \epsilon$ . It would suffice to have  $M|x_n| < \epsilon$  or  $|x_n| < \epsilon/M$ . Since  $\lim x_n = 0$ , there is  $N$  such that  $|x_n| < \epsilon/M$  for all  $n \geq M$ . Then  $|x_n y_n| < \epsilon$  for all  $n \geq M$  and  $\lim x_n y_n = 0$ .

5)(10pt) Suppose  $A \subset [0, 1]$  is infinite. Prove that there is  $\alpha \in [0, 1]$  a limit point of  $A$ .

Let  $a_1, a_2, a_3, \dots$  be distinct elements of  $A$ . This is a bounded sequence. By the Bolzano–Weierstrass Theorem,  $(a_n)_{n=1}^{\infty}$  has a convergent subsequence. That sequence converges to some  $\alpha$  a limit point of  $A$ . Since  $[0, 1]$  is closed  $\alpha \in [0, 1]$ .