1) (15pt) Give complete definitions of the following concepts:

a) \( f : A \rightarrow \mathbb{R} \) is continuous;

For all \( a \in A \) and for all \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( x \in A \) and \( |x - a| < \delta \), then \( |f(x) - f(a)| < \varepsilon \).

b) \( f : A \rightarrow \mathbb{R} \) is uniformly continuous;

For all \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( x, a \in A \) and \( |x - a| < \delta \), then \( |f(x) - f(a)| < \varepsilon \).

2) (15pt) Give complete statements of the following results:

a) Extreme Value Theorem

If \( K \) is compact and \( f : K \rightarrow \mathbb{R} \) is continuous, then \( f \) attains a maximum and minimum value.

b) Mean Value Theorem

If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous and \( f \) is differentiable on \((a, b)\), then there is \( a < \xi < b \) such that
\[
f'(\xi) = \frac{f(b) - f(a)}{b - a}.
\]

3) (30pt) Decide if the following statements are TRUE or FALSE. If FALSE, give an example showing it is FALSE.

a) If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous at every rational number, then \( f \) is continuous.

FALSE. For example
\[
f(x) = \begin{cases} 0 & \text{if } x < \pi \\ 1 & \text{if } x \geq \pi \end{cases}
\]
is continuous at all rationals but discontinuous at \( \pi \).

b) If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( A \subseteq \mathbb{R} \) is open, then \( f^{-1}(A) \) is open.

TRUE.

c) If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( A \) is compact, then \( f^{-1}(A) \) is compact.

FALSE. For example, let \( f(x) = \sin x \). Then \( f^{-1}([-1, 1]) = \mathbb{R} \).

d) There is a real number \( x \in [0, 1] \) such \( 3 - x^2 = e^x \).

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TRUE. (Let \( f(x) = 3 - x^2 - e^x \). Then \( f(0) = 2 \) while \( f(1) = 3 - 1 - e < 0 \). By the Intermediate Value Theorem there is \( x \in [0, 1] \) with \( f(x) = 0, e \) if \( f \) is differentiable at \( a \) and \( g \) is differentiable at \( f(a) \) then \( g \circ f \) is continuous at \( a \).

TRUE.

4)(15pt) Let \( f(x) = x^2 - 4x \). Using the definition of the derivative find \( f'(3) \).

\[
f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}
= \lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3}
= \lim_{x \to 3} \frac{(x - 1)(x - 3)}{x - 3}
= \lim_{x \to 3} x - 1
= 2.
\]

5)(15pt) Suppose \( f : [a, b] \to \mathbb{R} \) is differentiable and \( f'(x) \geq M \) for all \( x \in [a, b] \). Prove that \( f(b) \geq f(a) + M(b - a) \).

By the Mean Value Theorem, there is \( a < \xi < b \) such that

\[
f'(\xi) = \frac{f(b) - f(a)}{b - a}.
\]

Since \( f'(\xi) \geq M \),

\[
\frac{f(b) - f(a)}{b - a} \geq M
\]

and \( f(b) \geq f(a) + M(b - a) \).

6) (10pt) Suppose \( f : A \to \mathbb{R} \) is uniformly continuous and \( (a_n)_{n=1}^\infty \) is Cauchy where \( a_n \in A \) for all \( n \in \mathbb{N} \). Prove that the sequence \( (f(a_n))_{n=1}^\infty \) is Cauchy.

Let \( \epsilon > 0 \). There is \( \delta > 0 \) such that if \( x, y \in A \) and \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \). Since \( (a_n) \) is Cauchy, there is \( N \in \mathbb{N} \) such that \( |a_n - a_m| < \delta \) for all \( n, m \geq N \). But then \( |f(a_n) - f(a_m)| < \epsilon \) for all \( n, m \geq N \) and \( (f(a_n))_{n=1}^\infty \) is Cauchy.