Math 413 Analysis I

Midterm 2

November 26, 2003

1)(15pt) Give complete definitions of the following concepts:

a) $f : A \to \mathbb{R}$ is continuous;

For all $a \in A$ and for all $\epsilon > 0$ there is $\delta > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

b) $f : A \to \mathbb{R}$ is uniformly continuous;

For all $\epsilon > 0$ there is $\delta > 0$ such that if $x, a \in A$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

2) (15pt) Give complete statements of the following results:

a) Extreme Value Theorem

If K is compact and $f : K \to \mathbb{R}$ is continuous, then f attains a maximum and minimum value.

b) Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable on (a, b), then there is $a < \xi < b$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

3)(30pt) Decide if the following statements are TRUE or FALSE. If FALSE, give an example showing it is FALSE.

a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous at every rational number, then f is continuous.

FALSE. For example

$$f(x) = \begin{cases} 0 & \text{if } x < \pi \\ 1 & \text{if } x \geq \pi \end{cases}$$

is continuous at all rationals but discontinuous at π.

b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subseteq \mathbb{R}$ is open, then $f^{-1}(A)$ is open.

TRUE.

c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and A is compact, then $f^{-1}(A)$ is compact.

FALSE. For example, let $f(x) = \sin x$. Then $f^{-1}([-1, 1]) = \mathbb{R}$.

d) There is a real number $x \in [0, 1]$ such $3 - x^2 = e^x$.

TRUE. (Let $f(x) = 3 - x^2 - e^x$. Then $f(0) = 2$ while $f(1) = 3 - 1 - e < 0$. By the Intermediate Value Theorem there is $x \in [0, 1]$ with $f(x) = 0$, e) If f is differentiable at a and g is differentiable at $f(a)$ then $g \circ f$ is continuous at a.

TRUE.

4)(15pt) Let $f(x) = x^2 - 4x$. **Using the definition of the derivative** find $f'(3)$.

$$\begin{aligned} f'(3) &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3} \\ &= \lim_{x \to 3} \frac{(x-1)(x-3)}{x - 3} \\ &= \lim_{x \to 3} x - 1 \\ &= 2. \end{aligned}$$

5)(15pt) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable and $f'(x) \geq M$ for all $x \in [a, b]$. Prove that $f(b) \geq f(a) + M(b - a)$.

By the Mean Value Theorem, there is $a < \xi < b$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Since $f'(\xi) \geq M$,

$$\frac{f(b) - f(a)}{b - a} \geq M$$

and $f(b) \geq f(a) + M(b - a)$.

6) (10pt) Suppose $f : A \to \mathbb{R}$ is uniformly continuous and $(a_n)_{n=1}^{\infty}$ is Cauchy where $a_n \in A$ for all $n \in \mathbb{N}$. Prove that the sequence $(f(a_n))_{n=1}^{\infty}$ is Cauchy.

Let $\epsilon > 0$. There is $\delta > 0$ such that if $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Since (a_n) is Cauchy, there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \delta$ for all $n, m \geq N$. But then $|f(a_n) - f(a_m)| < \epsilon$ for all $n, m \geq N$ and $(f(a_n))_{n=1}^{\infty}$ is Cauchy.