Math 413 Analysis I
Problem Set 12

Due Friday November 21

1) Let $\epsilon > 0$. We say that $f : \mathbb{R} \to \mathbb{R}$ is ϵ-continuous at a if there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in V_\delta(a)$

a) Prove that f is continuous at a if and only if f is ϵ-continuous at a for all $\epsilon > 0$.

b) Let $D = \{a : f \text{ is not continuous at } a\}$. Let $D_\epsilon = \{a : f \text{ is not } \epsilon\text{-continuous at } a\}$. Prove that

$$D = \bigcup_{\epsilon > 0} D_\epsilon = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}.$$

2) Let $f : [0, +\infty)$ be the function $f(x) = \sqrt{x}$. Use the definition of the derivative to show that if $a > 0$, then f is differentiable at a and $f'(a) = \frac{1}{2\sqrt{a}}$.

3) Suppose $f, g, h : \mathbb{R} \to \mathbb{R}$ and $f(x) \leq g(x) \leq h(x)$ for all x, and $f'(a) = h'(a)$.

a) Suppose $f(a) = g(a) = h(a)$. Prove that g is differentiable at a and $g'(a) = f'(a) = h'(a)$.

b) Give an example showing this fails if we do not assume that $f(a) = g(a) = h(a)$.

4) Prove the following statement or give a counterexample. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is nonconstant, there is an $x \in \mathbb{R}$ such that $f'(x)$ is irrational.

5) a) Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous at 0. Let $f(x) = xg(x)$. Prove that f is differentiable at 0. What is $f'(0)$?

b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 0 and $f(0) = 0$. Prove that there is function $g(x)$ continuous at 0 such that $f(x) = xg(x)$ for all $x \in \mathbb{R}$. What is $g(0)$?