Quick Tour of the Topology of $\mathbb{R}$

Steven Hurder,
Dave Marker,
& John Wood [1]

[1]Department of Mathematics, University of Illinois at Chicago

April 17, 2003

Preface

These notes are a supplement for the "standard undergraduate course" in Analysis, Math 413 and 414, at the University of Illinois at Chicago. The aim is to present a more general perspective on the incipient ideas of topology encountered when exploring the rigorous theorem–proof approach to the results of Calculus. There is no attempt to make these notes comprehensive. Rather, the goal is to make them an extension of the course material, and to be read as such.

The "Three Hard Theorems" in Chapters 7 and 8 of M. Spivak's *Calculus* (listed as Theorems 1,2 and 3 in the Appendix) initiate the student to the ideas of topology. Sections 1–9 of these notes develop these ideas explicitly, and give the "topological" statements and proofs of the theorems.

In section 10, we discuss uniform continuity and give the proof that a continuous function on the closed interval is uniformly continuous (Theorem 4 in the Appendix). The proof is an immediate consequence of the compactness of the closed interval $[a, b]$. This material is contained in the Appendix to Chapter 8 of *Calculus*.

The highlight of the notes are a sequence of eaxamples and results, which are fundamentally deeper than the introductory ideas of the first ten sections. These ideas are all standard in more advanced textbooks on topology, but one of the ideas of these notes is that for $\mathbb{R}$, their proofs are readily accessible, and follow very naturally from the basic ideas already developed for the proofs of the basic theorems of topology in calculus. In particular, these notes include

- the construction of the Cantor Set;
- the construction of the Cantor Function;
- the characterization of which functions are Riemann integrable;
- a self-contained proof of the Baire Category Theorem for $\mathbb{R}$;
- the proof that the sets of points of continuity for a function are a G_δ set, hence cannot be the rational numbers;
- a proof of the existence of solutions to first order differential equations using the contraction mapping principle;
- a proof of the implicit function theorem using the contraction mapping principle.

Presenting these more advanced results in an easily accessible is our second goal, and hopefully the notes comes close to achieving this. We also hope that this broader perspective will be very useful when studying calculus in more general settings like

$\mathbb{R}^n$, and serve as a gentle introduction to a graduate course in real analysis and measure theory.

The general references at the end of the notes provide further reading for those interested in pursuing more of point-set topology.

CHAPTER 1

The Topology of $\mathbb{R}$

1. Open and Closed Sets

Definition 1.1 We say that $U \subseteq \mathbb{R}$ is *open* if for all $x \in U$, there is $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U$. That is, if $|x-y| < \epsilon$ then $y \in U$.

Intuitively, if U is open and $x \in U$, then every y that is sufficiently close to x is also in U.

For example, if $a < b$, then (a, b) is open. To show this, suppose that $x \in (a, b)$, and let $\epsilon \leq \min(x-a, b-x)$. If $|x-y| < \epsilon$, then $y \in (a, b)$, so $(x-\epsilon, x+\epsilon) \subset (a, b)$.

It is also easy to see that $\mathbb{R}$ is open, and that $(-\infty, a)$ and $(a, +\infty)$ are open for any $a \in \mathbb{R}$.

The emptyset $\emptyset$ is also open: since $\emptyset$ has no elements, it is vacuously true that every element of $\emptyset$ has a neighborhood contained in $\emptyset$.

Lemma 1.2. [**an** *arbitrary* **union of open sets is open**]
If U and V are open subsets of $\mathbb{R}$, then $U \cup V$ is open. More generally, if I is a set (it can even be an uncountable set) and $U_i \subseteq \mathbb{R}$ is open for each $i \in I$, then $W = \bigcup_{i \in I} U_i$ is open.

Proof: We prove the more general case. Suppose $x \in W = \bigcup_{i \in I} U_i$. Then there is $i \in I$ such that $x \in U_i$. Since U_i is open, there is an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U_i \subseteq W$. $\square$

The proof above does not use that I is a finite set – I can be infinite, or even uncountable – an *arbitrary* union of open sets is open.

For example $(n, n+1)$ is open for all $n \in \mathbb{Z}$. Thus

$$\mathbb{R} \setminus \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n+1)$$

is open.

Lemma 1.3. [**a finite intersection of open sets is open**]
Suppose U and V are open subsets of $\mathbb{R}$. Then $U \cap V$ is open. More generally, if $U_1, U_2, \ldots, U_n$ are open sets, then $U_1 \cap \cdots \cap U_n$ is open.

Proof: Suppose $x \in U \cap V$. Since U and V are open there are ϵ_1 and ϵ_2 such that $(x-\epsilon_1, x+\epsilon_1) \subseteq U$ and $(x-\epsilon_2, x+\epsilon_2) \subseteq V$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then $(x-\epsilon, x+\epsilon) \subseteq U \cap V$. The proof that $U_1 \cap \cdots \cap U_n$ is open follows similarly. □

It is not true that an arbitrary intersection of open sets is open. For example, let U_n be the interval $(-\frac{1}{n}, \frac{1}{n})$ for $n = 1, 2, \ldots$. Then $\bigcap_{n=1}^{\infty} U_n = \{0\}$ and $\{0\}$ is not open. The rule is that a *finite* intersection of open sets is open.

Definition 1.4 We say that $E \subseteq \mathbb{R}$ is *closed* if $\mathbb{R} - E$ is open.

Since $\mathbb{R}$ and $\emptyset$ are open, they are also both closed. If $a < b$, then $[a, b]$, $(-\infty, a]$ and $[b, +\infty)$ are closed.

LEMMA 1.5. [**an *arbitrary* intersection of closed sets is closed**]
If E and F are open subsets of $\mathbb{R}$, then $E \cap F$ is closed. More generally, if I is a set (it can even be an uncountable set) and $E_i \subseteq \mathbb{R}$ is closed for each $i \in I$, then $W = \bigcap_{i \in I} E_i$ is closed.

Proof:

$$\mathbb{R} - \bigcap_{i \in I} E_i = \bigcup_{i \in I} (\mathbb{R} - F_i)$$

Since each $\mathbb{R} - F_i$ is open, $\bigcap_{i \in I} F_i$ is closed by Lemma 1.2. □

LEMMA 1.6. [**a finite union of closed sets is closed**]
Suppose E and F are closed subsets of $\mathbb{R}$. Then $E \cup F$ is closed. More generally, if $E_1, E_2, \ldots, E_n$ are closed sets, then $E_1 \cup \cdots \cup E_n$ is closed.

Proof: E and F closed $\Longrightarrow \mathbb{R} \setminus E$ and $\mathbb{R} \setminus F$ open. By Lemma 1.3

$$\mathbb{R} \setminus (E \cup F) = (\mathbb{R} \setminus E) \cap (\mathbb{R} \setminus F)$$

is open, hence $E \cup F$ is closed. The proof that $E_1 \cup \cdots \cup E_n$ is closed is similar. □

LEMMA 1.7. *Suppose $E \subseteq \mathbb{R}$ is closed. If E is bounded above and $\beta = \sup E$, then $\beta \in E$. Similarly, if E bounded below and $\alpha = \inf E$, then $\alpha \in E$.*

Proof: Suppose for contradiction that $\beta \notin E$. Since $\mathbb{R} - E$ is open, there is $\epsilon > 0$ such that if $(\beta - \epsilon, \beta + \epsilon) \subset \mathbb{R} - E$, or $|x - \beta| < \epsilon \Longrightarrow x \notin E$. But then if $z \in (\beta - \epsilon, \beta)$ then z is also an upper bound for X, contradicting the fact that β is the least upper bound.

The proof that the greatest lower bound $\alpha \in E$ is similar. □

2. A topology for $\mathbb{R}$

A topology for a set X is a collection of subsets of X, $\mathcal{T} = \{U \mid U \in \mathcal{T}\}$ which satisfies three axioms:

- $\emptyset, X \in \mathcal{T}$ – the emptyset and the entire set belong to $\mathcal{T}$.
- If $\{U_i \mid i \in I\} \subset \mathcal{T}$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$ – the union of arbitrarily many sets in $\mathcal{T}$ belongs to $\mathcal{T}$.
- If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$ – the intersection of a finite numbers of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

We have seen one example of this – the open sets in $\mathbb{R}$ form a topology for $\mathbb{R}$. This is called the *metric topology*, because the open sets are defined using the distance function on $\mathbb{R}$. Based on this example, we say that a set $U \in \mathcal{T}$ is an *open set for $\mathcal{T}$*.

Can you construct another topology on $\mathbb{R}$? There are two other topologies on $\mathbb{R}$ whose definitions are "guess-able", and neither is the metric topology. That is, an "open set" fr the new topology need not be open in the ϵ sense – it need not contain an open interval about its points. (The idea of these other two topologies can be defined for any set X, not just for for $\mathbb{R}$.)

A set X with a topology $\mathcal{T}$ is called a topological set, or space. We denote this by $(X, \mathcal{T})$.

The idea of a topological space lets us define continuity of a function and convergence of a sequence without mentioning ϵ or δ anywhere. There are many important applications of this idea which extends the ϵ–δ method.

3. Continuous functions

Recall the definition of a continuous function $f \colon \mathbb{R} \to \mathbb{R}$:

Definition 3.1 [**ϵ-δ definition of continuous function**]
f is continuous at a if for all $\epsilon > 0$, there exists $\delta > 0$ so that

$$|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon$$

f is continuous (on $\mathbb{R}$) if it is continuous at every $a \in \mathbb{R}$.

This definition is good for some applications, but for others, it can be awkward, and hard to use in proofs. We show there is another, equivalent meaning to "f is continuous" using open sets.

Recall that if $f \colon \mathbb{R} \to \mathbb{R}$ and $X \subseteq \mathbb{R}$, then

$$f^{-1}(X) = \{a \in \mathbb{R} \colon f(a) \in X\}.$$

Proposition 3.2. **[open set definition of continuous function]**
Let $f\colon \mathbb{R} \to \mathbb{R}$. Then f is continuous if and only $f^{-1}(U)$ is open whenever $U \subseteq \mathbb{R}$ is open.

Proof: Suppose f is continuous. Let $U \subseteq \mathbb{R}$ be open and let $a \in f^{-1}(U)$. We need to show there is an interval around a contained in $f^{-1}(U)$. Now, $f(a) \in U$ and U is open, so there is $\epsilon > 0$ such that $(f(a)-\epsilon, f(a)+\epsilon) \subseteq U$. Since f is continuous there is $\delta > 0$ such that $|x-a| < \delta \implies |f(x) - f(a)| < \epsilon$. In particular, if $|x-a| < \delta$, then $f(x) \in U$. Thus $(a-\delta, a+\delta) \subseteq f^{-1}(U)$ which is what we wanted to show. This is true for every $a \in f^{-1}(U)$, so $f^{-1}(U)$ is open.

Conversely, let $a \in \mathbb{R}$ and $\epsilon > 0$ and set $U = (f(a)-\epsilon, f(a)+\epsilon)$. Since $f^{-1}(U)$ is open and $f(a) \in U$, we can find $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq f^{-1}(U)$. In other words, $|f(x) - f(a)| < \epsilon$ for all x such that $|x-a| < \delta$. Thus f is continuous as defined by Definition 3.1. □

Here is an application of this alternate definition:

Proposition 3.3. *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous functions. Then the composition $f \circ g\colon \mathbb{R} \to \mathbb{R}$ is continuous.*

Proof: Let $U \subset \mathbb{R}$ be open. Then $f \circ g^{-1}(U) = g^{-1}(f^{-1}(U))$.

Since f is continuous, the set $f^{-1}(U)$ is open.

Since g is continuous, the set $g^{-1}(f^{-1}(U))$ is open.

But this shows $f \circ g^{-1}(U)$ is open, which was to be proved. □

We can also define continuous function in terms of closed sets. On occasion, this alternate definition is more useful that the definition using open sets.

Lemma 3.4. **[closed set definition of continuous function]**
If $f\colon \mathbb{R} \to \mathbb{R}$, then f is continuous if and only if $f^{-1}(E)$ is closed for every closed $E \subseteq \mathbb{R}$.

Proof: First note that for any $Y \subseteq \mathbb{R}$

$$x \notin f^{-1}(Y) \Leftrightarrow f(x) \notin Y \Leftrightarrow x \in f^{-1}(\mathbb{R} \setminus Y).$$

We will use both directions of Proposition 3.2.

($\Rightarrow$) Suppose f is continuous and $X \subseteq \mathbb{R}$ is closed. Then $\mathbb{R} \setminus X$ is open and $\mathbb{R} \setminus f^{-1}(X) = f^{-1}(\mathbb{R} \setminus X)$ is open. Thus $f^{-1}(X)$ is closed.

($\Leftarrow$) Suppose $f^{-1}(X)$ is closed for every closed $X \subseteq \mathbb{R}$. Let $U \subseteq \mathbb{R}$ be open. Then $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus f^{-1}(U) = f^{-1}(\mathbb{R} \setminus U)$ are closed. Thus $f^{-1}(U)$ is open and f is continuous. □

Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. So $\mathcal{T}$ is a topology for X, and $\mathcal{S}$ is a topology for Y.

Definition 3.5 [general definition of continuous function]
A map $f\colon X \to Y$ is continuous (with respect to the topologies $\mathcal{T}, \mathcal{S}$) if for every $U \in \mathcal{S}$ the set $f^{-1}(U) \in \mathcal{T}$.

For the usual metric topology on $\mathbb{R}$, this definition is the above reformulation of the meaning of continuous. Here is an example of the more general notion of continuity:

Example 3.6 Let $X = \{f\colon [0,1] \to \mathbb{R}\}$ be the collection of *all functions* on the interval $[0,1]$. The set X can also be written in power notation as $X = \mathbb{R}^{[0,1]}$ – it is a very big set. The continuous functions $C[0,1] = \{f\colon [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ is a subset of X.

Given a function $f \in X$ and $\epsilon > 0$, define the subset of X

$$B(f,\epsilon) = \{g\colon [0,1] \to \mathbb{R} \mid \forall\, x \in [0,1], |g(x) - f(x)| < \epsilon\}$$

A set $U \subset X$ is open if

$$\forall\, f \in U,\ \exists\, \epsilon > 0 \text{ such that } B(f,\epsilon) \subset U$$

It is an exercise that this defines a topology for X, and also by restricting it, a topology on the set $C[0,1]$ of continuous functions. This is called the *uniform topology.*

Here is a nice exercise. Define a function

$$I\colon C[0,1] \to \mathbb{R},\ I(f) = \int_0^1 f(x)\,dx$$

Then $I\colon C[0,1] \to \mathbb{R}$ is a continuous function for the uniform topology.

4. Continuity and sequences

A sequence is a set function $a\colon \mathbb{N} \to \mathbb{R}$. It is usually written by listing its values $\{a_1, a_2, a_3, \ldots\}$, or just by $\{a_n\}$ for short where it is understood that the subscript n takes on the values of the positive integers. Sometime, a sequence will start at some subscript n which is not 1. For example, we write

$$\{a_n \mid n = 5, 6, 7, \ldots\} \;=\; \{a_5, a_6, a_7, \ldots\}$$

The set of values of a sequence $\{a_n\}$ define a subset $A \subset \mathbb{R}$ – this is the range of the function $a\colon \mathbb{N} \to \mathbb{R}$. But the sequence is different from its set of values, even though we write the sequence as $\{a_1, a_2, a_3, \ldots\}$ which looks like a set of values. For example, the set of values for the sequence $\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ is just the set $A = \{1, -1\}$.

We give two definitions of the limit of a sequence. The first uses ϵ–δ, while the second is in terms of open sets in the topology of $\mathbb{R}$. They are equivalent for $\mathbb{R}$, but the second definition is can be made to work for any topological space $(X, \mathcal{T})$, so we state a general third version also. In any case, we say that $\{a_n\}$ is a convergent sequence with limit L.

Definition 4.1 A sequence $\{a_n\}$ in $\mathbb{R}$ has a limit $L \in \mathbb{R}$, written $\lim_{n\to\infty} a_n = L$, or just $a_n \to L$, if

$$\forall\ \epsilon > 0,\ \exists\ N \text{ such that } n > N \Longrightarrow |a_n - L| < \epsilon$$

Definition 4.2 A sequence $\{a_n\}$ in $\mathbb{R}$ has limit $L \in \mathbb{R}$ if

$$U \subset \mathbb{R},\ L \in U\ \&\ U \text{ open},\ \exists\ N > 0, n > N \Longrightarrow a_n \in U$$

Definition 4.3 Let $(X, \mathcal{T})$ be a topological space. A sequence $\{a_n\} \subset X$ has limit $L \in X$ if

$$U \subset X,\ L \in U\ \&\ U \in \mathcal{T},\ \exists\ N > 0, n > N \Longrightarrow a_n \in U$$

A continuous function maps the points of a convergent sequence into the points of a convergence sequence:

THEOREM 4.4. *Let $\{a_n\}$ is a convergent sequence with limit L, and suppose $f(x)$ a continuous at $x = L$. Then the sequence $\{b_n = f(a_n)\}$ is convergent with limit $f(L)$.*

Proof: Given $\epsilon > 0$ we need to find N such that $n > N$ implies $|f(a_n) - f(L)| < \epsilon$. Since f is continuous at L, there exists $\delta > 0$ so that $|x - L| < \delta \Longrightarrow |f(x) - f(L)| < \epsilon$. Since $\lim_{n\to\infty} a_n = L$, for $\delta > 0$ given, there exists N so that $n > N \Longrightarrow |a_n - L| < \delta$. Then

$$n > N \Longrightarrow |a_n - L| < \delta \Longrightarrow |f(a_n) - f(L)| < \epsilon \Longleftrightarrow |b_n - f(L)| < \epsilon$$

□

There is a converse to this result, which often is the best (easiest) way to show a function is not continuous at some point:

THEOREM 4.5. *Suppose that $f(x)$ is defined on an interval (a, b) containing the point L, and for every convergent sequence $\{a_n\}$ a with limit L, the sequence $\{b_n = f(a_n)\}$ is convergent with limit $f(L)$. Then f is continuous at $x = L$*

Proof: We will prove the contra-positive: if $f(x)$ is not continuous at $x = L$, then there is a sequence $\{a_n\}$ such that $f(a_n)$ does not have limit $f(L)$. Assume $f(x)$ is not continuous at $x = L$. Then there exists some fixed $\epsilon > 0$ so that for every $\delta > 0$, there is some point $|x - L| < \delta$ with $|f(x) - f(L)| \geq \epsilon$. For each integer $n > 0$, set $\delta = 1/n$ and choose a point a_n so that $|a_n - L| < 1/n$ with $|f(a_n) - f(L)| \geq \epsilon$. . Then $|a_n - L| < 1/n$ but $|f(a_n) - L| \geq \epsilon$.

Claim: $\lim_{n\to\infty} a_n = L$

Given any $\delta > 0$ there exists some $N > 0$ such that $1/N < \delta$. Then for $n > N$,

$$|a_n - L| < 1/n \Longrightarrow |a_n - L| < 1/N \Longrightarrow |a_n - L| < \delta$$

So now we have produced a convergent sequence $\{a_n\}$ with limit L, yet the sequence $\{b_n = f(a_n)\}$ cannot have limit $f(L)$ as there exists an $\epsilon >$ so that for each n, $|b_n - L| \geq \epsilon$. $\square$

5. Connected Sets

Definition 5.1 We say that $X \subseteq \mathbb{R}$ is *connected* if for any open sets $U, V \subset \mathbb{R}$ such that

$$X \subseteq U \cup V,\ U \cap X \neq \emptyset, V \cap X \neq \emptyset \Longrightarrow U \cap V \neq \emptyset$$

That is, we cannot cover X by two disjoint open subsets of $\mathbb{R}$, both of which intersect X.

Definition 5.2 We say that $X \subseteq \mathbb{R}$ is *convex* if $a, b \in X \Longrightarrow [a, b] \subseteq X$.

For the line, these two notions are the same.

LEMMA 5.3. *Let $X \subseteq \mathbb{R}$. Then X is connected $\Longleftrightarrow$ X is convex.*

Proof: ($\Longrightarrow$) Suppose X is connected but not convex; we show this yields a contradiction. Then there are $a, b \in X$ with $[a, b] \not\subset X$, hence there is $a < c < b$ such that $c \notin X$. Let $U = \{x \in \mathbb{R} \mid x < c\}$ and $V = \{x \in \mathbb{R} \mid x > c\}$. Then U and V are disjoint open sets such that $a \in U$, $b \in V$ and $X \subseteq U \cup V$. Thus X in not connected.

($\Longleftarrow$) We give a proof by contradiction which uses the Intermediate Value Theorem.

Assume X is not connected, so there are open sets U and V such that

$$X \subseteq U \cup V,\ U \cap X \neq \emptyset,\ V \cap X \neq \emptyset,\ U \cap V = \emptyset$$

Suppose there are points $a, b \in X$ such that $a < b$, $a \in U$ and $b \in V$ (the reverse case $b < a$ is similar.) Since X is convex, $[a, b] \subset X$.

Let $f\colon [a, b] \to \mathbb{R}$ by $f(x) = \begin{cases} 0 & \text{if} \quad x \in U \\ 1 & \text{if} \quad x \in V \end{cases}$. Since $x \in [a, b] \subset X$ is in either U or V, but never both, f is a well defined function.

Claim: $f\colon [a, b] \to \mathbb{R}$ is continuous.

Let $c \in [a, b]$, then either $f(c) = 0$ or $f(c) = 1$, so we check each possibility. Suppose $f(c) = 0$, then $x \in U$ and there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq U$. Thus $f(x) = 0$

for all $x \in I \cap (c-\delta, c+\delta)$ and f is continuous at c. A similar argument shows that if $f(c) = 1$, then $c \in V$ and f is continuous at c. Thus f is continuous.

Since $f(a) = 0$ and $f(b) = 1$, by the Intermediate Value Theorem, there is c between a and b such that $f(c) = \frac{1}{2}$, but this is impossible, as the values of $f(x)$ are 0 or 1 only. $\square$

COROLLARY 5.4. *Suppose $a, b \in \mathbb{R}$ with $a < b$. The following sets are all connected: $\mathbb{R}$, $\{a\}$, $[a,b]$, (a,b), $(a,b]$, $[a,b)$, $(a,+\infty)$, $[a,+\infty)$, $(-\infty,a)$, and $(-\infty,a]$.*

Proof: All of these sets are convex. $\square$

The above proof that $X \subset \mathbb{R}$ convex implies X is connected is the simplest, but depends upon the Intermediate Value Theorem which is a deep consequence of the Least Upper Bound Axiom of the real numbers. It is also possible to prove Lemma 5.3 directly using the Least Upper Bound Axiom for the real numbers (see the proof of Theorem 6.2.8 in Sutherland) though the proof is not as simple. The advantage is that one can then use Lemma 5.3 to prove the Intermediate Value Theorem. We show how to relate these two ideas.

If $f\colon \mathbb{R} \to \mathbb{R}$, then the *image* of f is the set $f(X) = \{f(x) \mid x \in X\}$.

THEOREM 5.5. *If $f\colon \mathbb{R} \to \mathbb{R}$ and $X \subseteq \mathbb{R}$ is connected, then $f(X)$ is connected, and hence convex.*

Proof: Suppose that $f(X) \subset \mathbb{R}$ is not connected. Then there are open sets U and V such that $f(X) \subseteq U \cup V$, $f(X) \cap U \neq \emptyset$, $f(X) \cap V \neq \emptyset$ and $U \cap V = \emptyset$. By Proposition 3.2, $f^{-1}(U)$ and $f^{-1}(V)$ are open. It is easy to check that:

i) $X \subseteq f^{-1}(U) \cup f^{-1}(V)$,

ii) $f^{-1}(U) \cap X \neq \emptyset$ and $f^{-1}(V) \cap X \neq \emptyset$, and

iii) $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

This contradicts the fact that X is connected. $\square$

Theorem 5.5 is a generalization of the Intermediate Value Theorem. Suppose $f\colon [a,b] \to \mathbb{R}$ and $f(a) < 0 < f(b)$. By Theorem 5.5 $f([a,b])$ is connected, hence convex. Thus for all points z with $f(a) < z < f(b)$ there is some $c \in [a,b]$ such that $f(c) = z$. In particular, for $z = 0$, there must be some $c \in [a,b]$ with $f(c) = 0$. Since $f(a) < 0$ and $f(b) > 0$, $c \neq a, b$ so $a < c < b$. $\square$

6. Compact Sets

The idea of a compact set is maybe one of the ideas that separates "Analysis" from "Calculus". It is a fundamental concept, which gets used in many ways. One principle is that "a continuous function on a compact set is akin to an arbitrary function on a finite set." The ideas in this section appear repeatedly in proofs in analysis, usually in different chapters or contexts. Compactness is the common concept which has been extracted. Think of it as the "vanilla" that gets added to many proofs to give them zest!

Definition 6.1 Let $(X, \mathcal{T})$ be a topological space. An open cover for a subset $E \subset X$ is a collection of open sets $\{U_i \subset X \mid i \in I\}$ so that $E \subseteq \bigcup_{i \in I} U_i$.

The open sets U_i do not have to be contained in E, and usually are not. The index set I can be any set, countable or uncountably infinite. If I is finite (resp. countable), then $\{U_i \subset X \mid i \in I\}$ is called a finite (resp. countable) open cover of E.

Definition 6.2 Let $(X, \mathcal{T})$ be a topological space. We say that $E \subseteq X$ is *compact* if whenever $\{U_i \mid i \in I\}$ is an open cover of E, there is a finite subset $I_0 \subseteq I$ such that $\{U_i \mid i \in I_0\}$ is also an open cover of E.

In other words, E is compact if *every open cover has a finite subcover.*

These are general definitions - we now consider only the case $X = \mathbb{R}$, where compactness has a much more concrete description. First, an open set $U \subset \mathbb{R}$ is always countable union of intervals. An open cover of a set E is then a collection of open sets U whose union contains E.

Here is one of the basic examples. One way to get an open cover of a set E is to chose for each $x \in E$ some $\epsilon_x > 0$, then $U_x = (x - \epsilon_x, x + \epsilon_x)$ is open in $\mathbb{R}$, and the collection $\{U_x \mid x \in E\}$ contains E in its union – the index set $I = E$ itself. The size of the intervals does not matter, since $x \in (x - \epsilon_x, x + \epsilon_x)$ no matter what $\epsilon_x > 0$ is. If E is compact, then any such collection of open sets must contain a finite subcover which also contains E. That is, if E is compact, there must be points $\{x_1, x_2, \ldots, x_n\} \subset E$ such that

$$E \subset U_{x-1} \cup \cdots \cup U_{x_n} = (x_1 - \epsilon_1, x_1 + \epsilon_1) \cup \cdots \cup (x_n - \epsilon_n, x_n + \epsilon_n)$$

The amazing fact about $\mathbb{R}$ is that $E \subset \mathbb{R}$ is compact $\iff$ E is closed and bounded. We prove this in three steps.

LEMMA 6.3. *$E \subseteq \mathbb{R}$ compact $\implies$ E bounded, i.e. there is a number M such that $|x| < M$ for all $x \in E$.*

Proof: For each $n \in \mathbb{N}$, let $U_n = (-n, n)$. Then $\{U_n \mid n \in \mathbb{N}\}$ is an open cover of E (the union is $\mathbb{R}$, so contains E). E compact implies there is a finite subset of integers

$\{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}$ so that $E \subseteq \bigcup_{i=1}^{k} U_{n_i}$. Set $M = \max\{n_1, \ldots, n_k\}$, the largest of the finite set of integers. The union of the $(-n_i, n_i)$ is just the interval $(-M, M)$ hence

$$x \in E \Longrightarrow x \in (-M, M) \Longrightarrow |x| < M \quad \square$$

Lemma 6.4. *$E \subseteq \mathbb{R}$ compact $\Longrightarrow E$ closed.*

Proof: We will show that $\mathbb{R} - E$ is open. Suppose $a \notin E$. For all $n \in \mathbb{N}$ with $n > 0$, let

$$U_n = \{x \mid |x - a| > \frac{1}{n}\} = (-\infty, a - \frac{1}{n}) \cup (a + \frac{1}{n}, +\infty)$$

Then U_n is open and $\bigcup_{n=1}^{\infty} U_n = \mathbb{R} - \{a\} \supset E$. Thus $\{U_n \mid n > 0\}$ is an open cover of E. Since E is compact, there is a number N such that

$$E \subseteq \bigcup_{n=1}^{N} U_n = (-\infty, a - \frac{1}{N}) \cup (a + \frac{1}{N}, +\infty)$$

Choose $\epsilon > 0$ such that $0 < \epsilon < \frac{1}{N}$. Then

$$(a - \epsilon, a + \epsilon) \cap E \subset (a - \epsilon, a + \epsilon) \cap \left((-\infty, a - \frac{1}{N}) \cup (a + \frac{1}{N}, +\infty) \right) = \emptyset$$

so $(a - \epsilon, a + \epsilon) \subseteq \mathbb{R} - E$. This shows that $\mathbb{R} - E$ is open, hence E is closed. $\square$

Together, these two lemmas show $E \subseteq \mathbb{R}$ compact $\Longrightarrow E$ closed and bounded. We need to prove the converse is true; the first step is to show that a closed and bounded *interval* is compact.

Theorem 6.5. **[Heine-Borel]** *The closed interval $[a, b]$ is compact.*

Proof: Suppose $\{U_i \mid i \in I\}$ is an open cover of $[a, b]$. The idea is to define a the largest sub-interval of $[a, b]$ which does have a finite subcover, then prove this subinterval is really all of $[a, b]$. Let

$$A = \left\{ x \in [a, b] \mid \text{ there is } I_0 \subseteq I \text{ finite such that } [a, x] \subseteq \bigcup_{i \in I_0} U_i \right\}$$

The initial point $a \in A$ is contained in some open set U_{i_0} for some $i_0 \in I$, so $\{a\}$ has an open covering with one set, $\{U_{i_0}\}$. Thus, $a \in A$ and $A \neq \emptyset$.

Every element of A is bounded above by b, so A is non-empty and bounded above, hence there is a least upper bound $d = \sup A$.

Claim: *$d > a$.*

Since U_{i_0} is open, there is an $\epsilon > 0$ such that $(a, a+\epsilon) \subseteq U_{i_0}$. Then $\{U_{i_0}\}$ is an open cover for the interval $(a, a+\epsilon)$, hence $(a, a+\epsilon) \subseteq A$. This shows so $\sup A \geq a+\epsilon > a$. □

Claim: $d = b$.

Suppose that $d < b$. Since $\{U_i \mid i \in I\}$ is an open cover of $[a, b]$, there is $\ell \in I$ such that $d \in U_\ell$. Since U_ℓ is open, there is $\epsilon > 0$ such that $(d-\epsilon, d+\epsilon) \subseteq U_\ell$. Since d is the least upper bound for A, we can find c such that $d - \epsilon < c \leq d$ and $c \in A$.

By definition, $c \in A$ implies there is a finite subset $\{j_1, \ldots, j_m\} \subset I$ such that $[a, c] \subseteq U_{j_1} \cup \cdots \cup U_{j_m}$.

Since we assume $d < b$, there is $z < b$ such that $d < z < d + \epsilon$. Then

$$[a, z] \subseteq U_{j_1} \cup \cdots \cup U_{j_m} \cup U_\ell$$

This implies that $z \in A$. This contradicts the fact that d is the least upper bound for A. □

Given that $\sup A = b$, we need to show there is a finite subcover of $[a, b]$. Let $k \in I$ such that $b \in U_k$. There is some $\epsilon > 0$ such that $(b - \epsilon, b + \epsilon) \subset U_k$. Choose a point $z < b$ with $b - \epsilon < z < b$. Then $b = \sup A$ implies $z \in A$ which implies there is a finite subset $\{j_1, \ldots, j_m\} \subset I$ such that $[a, z] \subseteq U_{j_1} \cup \cdots \cup U_{j_m}$. The collection $\{U_{j_1}, \ldots, U_{j_m}, U_k\}$ is then a finite open cover for $[a, b]$. □

We combine the above three results to prove one of the main theorems of the topology of the line.

THEOREM 6.6. *For $E \subseteq \mathbb{R}$, E compact $\iff$ E closed and bounded.*

Proof: If E is compact, then E is bounded by Lemma 6.3, and E is closed by Lemma 6.4.

Assume that E is closed and bounded. E bounded means we can find M such that $X \subseteq [-M, M]$. We use the fact that $[-M, M]$ is compact to deduce that E is compact.

Let $\{U_i \mid i \in I\}$ be an open cover of E. We need to find a finite subcover. E is closed implies $V = \mathbb{R} - E$ is open, so $\{U_i \mid i \in I\} \cup \{V\}$ is an open cover of $[-M, M]$. (A point is either in E or in its complement V, so is either in the union of the cover of E or in V.) Since $[-M, M]$ is compact, there is a finite subset $\{i_1, \ldots, i_m\} \subset I$ such that

$$[-M, M] \subseteq U_{i_1} \cup \ldots U_{i_m} \cup V$$

Then $E \subset [-M, M]$ and $E \cap V = \emptyset$ implies $E \subseteq U_{i_1} \cup \ldots U_{i_m}$.

We have produced a finite subcover of the given open cover, as claimed. □

COROLLARY 6.7. *If $X \subset \mathbb{R}$ is compact and $Y \subseteq X$ is closed, then Y is compact.*

Proof: Since X is bounded, Y is bounded. Since Y is closed and bounded it is compact. □

There is a very general version of this Theorem, which applies to any topological space $(X, \mathcal{T})$ where there is a distance function on X so that a set U is open exactly when it is a union of ϵ-balls, where the ϵ varies with each set. $(X, \mathcal{T})$ is called metric topological space. The proof of the following is part of every course in point set topology.

THEOREM 6.8. *Let $(X, \mathcal{T})$ be a metric topological space. $E \subset X$ is compact $\iff E$ is closed and bounded.*

7. Sequential compactness

The definition of a compact set is novel, and it takes a while to get an intuition for coverings by open sets. But there is another version of compactness for sets in $\mathbb{R}$ which is much more easy to understand, called *sequential compactness*. We discuss only the case where $X = \mathbb{R}$ – though all of the following holds for metric topological spaces.

Recall that a sequence $\{a_1, a_2, a_3, \dots\}$ has a limit L if

$$\forall\ \epsilon > 0,\ \exists N \text{ such that } n > N \Longrightarrow |a_n - L| < \epsilon$$

A subsequence is a new sequence formed from $\{a_1, a_2, a_3, \dots\}$ by choosing a monotone increasing subset of indices, $\{n_1 < n_2 < \cdots\}$, then the subsequence is $\{a_{n_1}, a_{n_2}, a_{n_3}, \dots\}$. For example, we get a subsequence by choosing only the odd subscripts $\{a_1, a_3, a_5, \dots\}$. If necessary to make things clear, you can "re-label" the sequence $b_k = a_{n_k}$ then the subsequence is $\{b_1, b_2, b_3, \dots\}$.

The subsequence does not have to be given by any particular rule, or have any "regularity". For example, if we take the index function $n_k = k!$ then the terms of the subsequence are terms in $\{a_{k!}\}$ which are further and further spaced apart: $\{a_1, a_2, a_6, a_{24}, a_{120}, \dots\}$. Or, just choose a random sequence of *increasing* integers for n_k to get a random subsequence $\{a_{n_k}\}$.

A sequence $\{a_n\}$ is said to have a *convergent subsequence* if there is a subsequence $\{a_{n_k}\}$ with $\lim_{k\to\infty} a_{n_k} = L$.

For example, the sequence $\{a_n = n\}$ has no convergent subsequence, as every infinite subsequence contains points which tend to infinity. On the other hand, for the sequence $\{a_n = (-1)^n\}$, both the odd and the even subsequences are convergent, though to different limits. Here is a hard exercise: use the irrationality of π to prove that the sequence $\{a_n = \sin(n)\}$ has a convergent subsequence.

Definition 7.1 A subset $E \subset \mathbb{R}$ is *sequentially compact* if every infinite sequence $\{a_n\} \subset E$ has a convergent subsequence $\{a_{n_k} \mid k = 1, 2, \dots\}$ with $\lim_{k\to\infty} a_{n_k} = L \in E$.

It is important part of the definition that the limit $L \in E$. Otherwise, the set $(0, 1)$ would be sequentially compact, for example. Here is the main result, which implies that a subset $E \subset \mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

THEOREM 7.2. *For $E \subset \mathbb{R}$, E compact $\Longleftrightarrow$ E sequentially compact.*

Proof: ($\Longrightarrow$) Assume that E is compact, but there exists an infinite sequence $\{a_n\} \subset E$ with no convergent subsequence in E. We will show this implies that E is not compact, a contradiction.

First, suppose that the set of values $A = \bigcup_{n=1}^{\infty} \{a_n\}$ of the sequence is a finite set, say $\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subset E$. Since the sequence $\{a_n\}$ is infinite, there must exists some index $1 \leq i \leq m$ for which $a_n = \alpha_i$ for an infinite number of n. Then there is an infinite subsequence which is constant, $a_{n_k} = \alpha_i$ for all k. So we have $a_{n_k} \to \alpha_i \in E$. This contradicts the assumption there is no convergent subsequence to $L \in E$. Thus, we can assume that the set of values A of the sequence is an infinite subset of E.

Next, suppose that for some $x \in E$, for every $\epsilon > 0$ the intersection $(x - \epsilon, x + \epsilon) \cap A$ is an infinite set. We can then define inductively a a convergent subsequence of $\{a_n\}$. For each positive integer n, take $\epsilon = 1/n$. Chose n_1 such that $a_{n_1} \in (x - 1, x + 1) \cap A$. Assume that $n_1 < n_2 < \cdots n_{k-1}$ have been chosen. Since $(x - 1/k, x + 1/k) \cap A$ is an infinite set, we can choose $n_k > n_{k-1}$ with $a_{n_k} \in (x - 1/k, x + 1/k) \cap A$. It is then clear that $\lim_{k \to \infty} a_{n_k} = x \in E$.

Since we assume $\{a_n\} \subset E$ is an infinite sequence with no convergent subsequence in E, the above argument shows that for each $x \in E$, there is some $\epsilon_x > 0$ so that $(x - \epsilon_x, x + \epsilon_x)$ contains only a finitely many terms of the sequence $\{a_n\}$. Hence, $A \cap (x - \epsilon_x, x + \epsilon_x)$ is a finite set. Let $U_x = (x - \epsilon_x, x + \epsilon_x)$ be the open interval centered at x. Then the collection of open sets $\{U_x \mid x \in E\}$ is an open covering of E.

As E is compact, there exists points $\{x_1, x_2, \ldots, x_n\} \subset E$ such that

$$E \subset U_{x-1} \cup \cdots \cup U_{x_n} = (x_1 - \epsilon_1, x_1 + \epsilon_1) \cup \cdots \cup (x_n - \epsilon_n, x_n + \epsilon_n)$$

This implies that $A = A \cap E = (A \cap U_{x_1}) \cup \cdots \cup (A \cap U_{x_k})$.

We are given that the set of values $A \subset E$ is infinite. Yet, each of the sets $A \cap U_{x_i}$ for $1 \leq i \leq k$ is finite by our previous argument, which implies that the set of values A for the sequence $\{a_n\}$ is a finite union of finite sets, hence is finite. This is a contradiction.

($\Longleftarrow$) We show that if E is not closed, then E is not sequentially compact, and if E is not bounded then E is not sequentially compact. Thus, the hypotheses E is sequentially compact implies E is closed and bounded hence E is compact by Theorem 6.6.

E not closed implies $\mathbb{R}-E$ is not open, so there exists a point $x \in \mathbb{R}-E$ so that every ϵ interval about x is not contained in $\mathbb{R}-E$. That is, each interval $(x-\epsilon, x+\epsilon) \cap E \neq \emptyset$. For $\epsilon = 1/n$ let

$$a_n \in (x - 1/n, x + 1/n) \cap E$$

then $a_n \to x \in \mathbb{R} - E$, so E is not sequentially compact.

E not bounded implies for every $n > 0$ there exists some point $a_n \in E$ with $|a_n| > n$. Otherwise, if there is an n with $E \subset [-n, n]$ and so E would be bounded. Then the sequence $\{a_n\}$ has no convergent subsequence as $|a_n| \to \infty$. □

8. Nested Interval Theorem

We give two proofs of the following important result. The first uses the Least Upper Bound Axiom of $\mathbb{R}$, while the second is more simple, and is based on the Heine-Borel Theorem.

THEOREM 8.1. **[Nested Interval Theorem]**
Suppose that $I_n = [a_n, b_n]$ is a non-empty closed interval for all n, and satisfies $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

The condition on the intervals is that they are all "nested", one inside the other. The conclusion is that there must be some point which lies in all of them - the infinite intersection is non-empty.

Proof: If $n > m$, $[a_n, b_n] \subset [a_m, b_m]$, and so $a_m \leq a_n \leq b_n \leq b_m$. This implies that the sequence of lower endpoints $\{a_n\}$ is non-decreasing.

Also, $n > m \implies a_m \leq a_n \leq b_n \leq b_m$ implies $a_n \leq b_m$ for all n, m. So, the set $\{a_n\}$ is bounded above by b_m for all m, hence has a least upper bound $\alpha = \lim_{n\to\infty} a_n$. Moreover, each b_m is an upper bound for the sequence $\{a_n\}$, so $\alpha \leq b_m$ as α is the least upper bound.

Similarly, the sequence of upper endpoints $\{b_m\}$ is non-increasing, and $a_n \leq b_m$ for all n, m. So, the set $\{b_m\}$ is bounded below by a_n for all n, hence has a greatest lower bound $\beta = \lim_{m\to\infty} b_m$. Moreover, $\alpha \leq b_m$ for all m implies $\alpha \leq \beta$.

We thus have $a_n \leq \alpha \leq \beta \leq b_n$ for all n, or $[\alpha, \beta] \subset [a_n, b_n]$ for all n. Thus, $[\alpha, \beta] \subset \bigcap_{n=1}^{\infty} I_n$ and as $\alpha \leq \beta$ the set $[\alpha, \beta]$ is not empty. □

Here is a second proof, which uses the compactness of closed intervals.

Proof: The interval $[a_1, b_1]$ is closed and bounded, hence compact. The idea of the proof is to assume that the infinite intersection is empty, then we produce an infinite open cover of $[a_1, b_1]$ which must then have a finite subcover and show this gives a contradiction.

For each n define an open set $U_n = (-\infty, a_n) \cup (b_n \infty) = \mathbb{R} - [a_n, b_n]$. Note that $I_n \subset I_m \implies U_m \subset U_n$ - the reverse inclusion. Also,

$$\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (\mathbb{R} - [a_n, b_n]) = \mathbb{R} - \bigcap_{n=1}^{\infty} [a_n, b_n]$$

Now assume $\bigcap_{n=1}^{\infty} [a_n, b_n] = \emptyset$. Then the union $\bigcup_{n=1}^{\infty} U_n = \mathbb{R}$. Consequently, the collection $\{U_n \mid n = 1, 2, \dots\}$ is an open cover of $[a_1, b_1]$. Therefore, there is a finite subcover $\{U_{n_1}, \dots, U_{n_k}\}$. We re-order the subscripts so they are increasing, $n_1 < n_2 < \cdots < n_k$, then $U_{n_1} \subset U_{n_2} \subset \cdots \subset U_{n_k}$, so the union is just the last open set U_{n_k}. This means

$$[a_1, b_1] \subset U_{n_1} \cup \cdots \cup U_{n_k} = U_{n_k} = (-\infty, a_{n_k}) \cup (b_{n_k}, \infty)$$

This leads to a contradiction, as it implies $[a_{n_k}, b_{n_k}] \subset [a_1, b_1] \subset \mathbb{R} - [a_{n_k}, b_{n_k}]$. $\square$

The second proof is somewhat simpler than the first proof, but it has another, more important advantage. The first proof uses the ordering of the line and the least upper bound property of $\mathbb{R}$. The second proof uses only the compactness property of the initial set $[a_1, b_1]$.

Georg Cantor gave the most general version of this theorem for $\mathbb{R}$ more than a hundred years ago:

THEOREM 8.2. **[Nested Compact Set Theorem]**
Suppose that $E_n \subset \mathbb{R}$ is a non-empty compact set for all n, and satisfies $E_1 \supset E_2 \supset \cdots$. Then $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$.

In this form, the result is true if we just assume that E_n is a descending chain of closed subsets of a compact set E_1 in a a complete metric space. For example, the Nested Compact Set Theorem is true for descending chains of closed and bounded subsets of $\mathbb{R}^m$.

9. Continuous functions on compact sets

Continuous functions on compact sets have many special properties. The first property is a general version of the maximum/minimum principle for continuous functions on a closed interval $[a, b]$.

THEOREM 9.1. *If $E \subseteq \mathbb{R}$ is continuous and $f \colon \mathbb{R} \to \mathbb{R}$ is continuous, then $f(E)$ is compact*

Proof: Suppose $\{U_i \mid i \in I\}$ is an open cover of $f(X)$. Then f continuous implies $f^{-1}(U_i)$ is open for all $i \in I$. The union of the open sets covers the image $f(X)$ hence

$$X \subseteq \bigcup_{i \in I} f^{-1}(U_i)$$

This shows the collection $\{f^{-1}(U_i) \mid i \in I\}$ is an open cover of X.

Since X is compact, there is a finite $I_0 \subseteq I$ such that $X \subseteq \bigcup_{i \in I_0} f^{-1}(U_i)$ and so $f(X) \subseteq \bigcup_{i \in I_0} U_i$. Thus $\{U_i \mid i \in I_0\}$ is a finite subcover of $f(X)$. Given an open covering $\{U_i \mid i \in I\}$ of $f(X)$, we have shown there exists a finite subcovering. Hence, $f(X)$ is compact. □

Two of the "Three Hard Theorems" in Chapter 7, 8 of the text by Spivak were about finding the max and min of a continuous function on a closed interval $[a, b]$. Here is the generalized version of these results using compactness.

Corollary 9.2. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $E \subseteq \mathbb{R}$ is compact, then there are $c, d \in E$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in E$.*

Proof: By Theorem 9.1, $f(X)$ is compact.

By the Heine-Borel Theorem, $f(X)$ is closed and bounded.

By Lemma 1.7 there is $d \in X$ such that $f(d)$ is the least upper bound of $f(X)$. Similarly, there is $c \in X$ such that $f(c)$ is the greatest lower bound of $f(X)$. Thus, $f(X) \subset [f(c), f(d)]$. That is, for all $x \in X$, $f(c) \leq f(x) \leq f(d)$. □

Corollary 9.3. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $E = [a, b]$, then there are $a \leq c, d \leq b$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in E$.*

Proof: The closed interval $[a, b]$ is compact by Theorem 6.5 □

10. Uniform Continuity

The ϵ–δ Definition 3.1 that f is continuous on a set $X \subset \mathbb{R}$ states

$$\forall x \in X, \forall \epsilon > 0, \ \exists\, \delta > 0, \text{ such that } |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon$$

In other words, the existence of δ is asserted after being given x and $\epsilon > 0$. It can happen that $\delta > 0$ depends on which x is given – so the choice of δ is not *uniform* in x.

The idea of uniform continuity is to demand that the choice of δ is independent of the choice of $x \in X$. This is formulated by saying that the choice of δ precedes the choice of x as here.

Definition 10.1 $f\colon \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* on $X \subset \mathbb{R}$ if

$$\forall \epsilon > 0,\ \exists\, \delta > 0,\ \text{such that } \forall x \in X,\ |x-a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon$$

For example, it is proved in analysis texts that a continuous function $f\colon [a,b] \to \mathbb{R}$ is uniformly continuous. (For example, see the Appendix to Chapter 8 of Spivak's *Calculus*.) The usual proof is a least upper bound construction which is essentially the same as the proof of the Heine-Borel Theorem 6.5. Thus, it is no surprise that compactness and uniform continuity are related.

THEOREM 10.2. *If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $X \subseteq \mathbb{R}$ is compact, then f is uniformly continuous on X.*

Proof: Let $\epsilon > 0$. For each $a \in X$ there is δ_a such that

$$|x-a| < \delta_a \Longrightarrow |f(x) - f(a)| < \epsilon/2$$

Let $U_a = (a - \delta_a/2, a + \delta_a/2)$. Then $\{U_a \mid a \in X\}$ is an open cover of X. Since X is compact there are $a_1, \ldots, a_m \in X$ such that $U_{a_1}, \ldots, U_{a_m}$ is an open cover of X. Let $\delta = \min\{\delta_{a_1}/2, \ldots, \delta_{a_m}/2\}$.

Suppose $b \in X$ and $|x-b| < \delta$. There is an a_i such that $b \in O_{a_i}$. Then $|b-a| < \delta_{a_i}/2$ and $|f(b) - f(a)| < \epsilon/2$. Since $|x-b| < \delta \leq \delta_{a_i}/2$ and $|b - a_i| < \delta_{a_i}/2$, $|x-a_i| < \delta_{a_i}$ and $|f(x) - f(a_i)| < \epsilon/2$. Thus

$$|f(x) - f(b)| < |f(x) - f(a_i)| + |f(a_i) - f(b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and f is uniformly continuous on X. $\square$

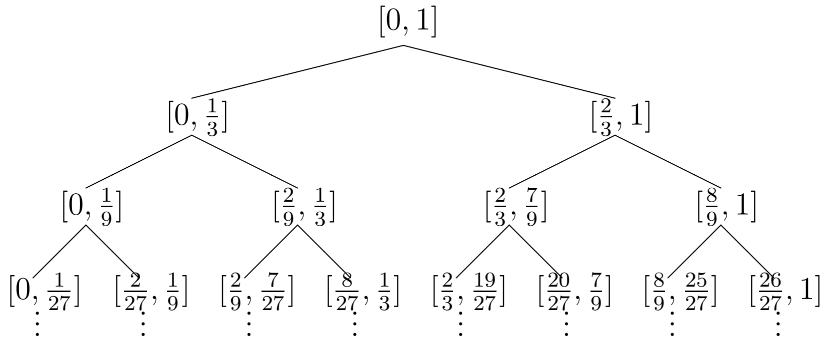
COROLLARY 10.3. *If $f\colon [a,b] \to \mathbb{R}$ is continuous, then f is uniformly continuous on $[a,b]$.*

The uniform continuity of $f\colon [a,b] \to \mathbb{R}$ is used to give a simple proof that f is Riemann integrable on $[a,b]$.

11. The Cantor Set

The Cantor set $\mathbb{X}$ is a compact set that does not look anything like an interval. It is an important example for many ideas of topology of $\mathbb{R}$.

This set $\mathbb{X}$ is built by induction. We start with the unit interval $[0,1]$ and throw out the middle third $(\frac{1}{3}, \frac{2}{3})$. This leaves two closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. We next throw out the middle third of each of those intervals and are left with four closed intervals. We continue this way. We build a tree of intervals that looks like this:



This is also called the "middle third" Cantor set. We describe the construction more precisely as follows.

Let S_n be all sequences of zeros and ones of length n. A typical $\sigma \in S_n$ is a finite sequence of zeros and ones of length n, $\sigma = \{i_1, i_2, \dots, i_n\}$. The set $S_0 = \emptyset$; the set S_1 has two elements, $\{0\}$ and $\{1\}$; the set S_2 has four elements, $\{0, 0\}$, $\{1, 0\}$, $\{1, 0\}$, $\{1, 1\}$. In general, S_n has 2^n elements.

For each $\sigma \in S_n$, we define inductively an interval $I_\sigma = I_{i_1,i_2,\dots,i_n}$ of length $\dfrac{1}{3^n}$ as follows. For $n = 0$, $S_0 = \emptyset$ the empty sequence. The corresponding interval is $I_\emptyset = [0, 1]$.

Now, suppose that for $\sigma \in S_n$ we have defined $I_\sigma = [a, b]$, then there are two sequences $\tau \in S_{n+1}$ of length $n + 1$ with initial segment σ, $\tau = \{\sigma, 0\}$ and $\tau = \{\sigma, 1\}$. Then set

$$I_{\sigma,0} = [a, \frac{b-a}{3}] \ \&\ I_{\sigma,1} = [\frac{2(b-a)}{3}, b]$$

Note that if σ is an initial segment of τ, then $I_\tau \subseteq I_\sigma$. That is, if $\sigma = \{i_1, i_2, \dots, i_n\}$ and $\sigma = \{i_1, i_2, \dots, i_n, \dots, i_m\}$, then $I_{i_1,i_2,\dots,i_m} \subseteq I_{i_1,i_2,\dots,i_n}$.

For each $n > 0$, let $\mathbb{X}_n = \bigcup_{\sigma \in S_n} I_\sigma$. The set $\mathbb{X}_n$ is the union of 2^n closed intervals, so is a closed set.

Let U_n be the set of points in $[0, 1]$ that are "thrown out" by stage n: $U_n = [0, 1] - \mathbb{X}_n$. There are 2^n sequences in S_n, including the endpoints of $[0, 1]$, so U_n is the union of $2^n - 1$ open intervals. Moreover, the set $V_n = U_m - U_{n-1}$ is the union of 2^{n-1} disjoint open intervals, each of length 3^{-n}. For later use, note that the 2^{n-1} intervals in the set V_n can be labeled by their order in the interval $[0, 1]$,

$$V_n = V_{n,1} \cup \cdots \cup V_{n,2^{n-1}}$$

Let $U = \bigcup_{n>0} U_n$ be the open set which is the union of all the open intervals $V_{n,\ell}$, $n > 0$ and $1 \leq \ell < 2^n$.

Definition 11.1 The *Cantor set* is $\mathbb{X} = [0,1] - U = \bigcap_{n>0} \mathbb{X}_n$.

Here is the picture of the complements $[0,1], \mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ and $\mathbb{X}_4$

What is left in the infinite intersection $\mathbb{X}$ is called "Cantor's Dust" as it looks like just a cloud of points.

It is easy to see that $0, 1 \in \mathbb{X}$, as these points lie in every $\mathbb{X}_n$. What other numbers are in $\mathbb{X}$? Let S_∞ be the set of all functions from $\mathbb{N}$ to $\{0,1\}$. That is, $\sigma \in S_\infty$ corresponds to an infinite sequence $\{i_1, i_2, \dots\}$ where each $i_n \in \{0,1\}$. Let $\sigma|n$ be the finite sequence $\{i_1, \dots, i_n\}$. Then we get an infinite sequence of nested closed intervals,

$$I_{\sigma|0} \supset I_{\sigma|1} \supset I_{\sigma|2} \supset \dots$$

where each $I_{\sigma|n}$ has length 3^{-n}. By the Nested Interval Theorem 8.1, there is a unique real number $x_\sigma \in [0,1]$ such that $x_\sigma = \bigcap_{n=1}^{\infty} I_{\sigma|n}$. Since $x_\sigma \in I_{\sigma|n}$ for all n, $x_\sigma \in \mathbb{X}_n$ for all n, thus $x_\sigma \in \mathbb{X}$.

PROPOSITION 11.2. *Let $G\colon S_\infty \to \mathbb{X}$ by $G(\sigma) = x_\sigma$. Then G is a one to one map from S_∞ onto $\mathbb{X}$.*

Proof: Suppose $\sigma, \tau \in S_\infty$ and $\sigma \neq \tau$. Let n be least such that $\sigma(n) \neq \tau(n)$. Then $I_{\sigma|n} \cap I_{\tau|n} = \emptyset$. Since $x_\sigma \in I_{\sigma|n}$ and $x_\tau \in I_{\tau|n}$, $x_\sigma \neq x_\tau$. Thus G is one to one.

Suppose $x \in \mathbb{X}$. We inductively define $\sigma \in S_\infty$ such that $x = x_\sigma$. Suppose we know $\sigma(1), \dots, \sigma(n)$ and $x \in I_{\sigma|n}$. Since $x \in \mathbb{X}$, $x \in I_{\sigma|n,i}$ for $i = 0$ or 1. Let $\sigma(n+1) = i$. □

The set S_∞ is bijective with the set $2^{\mathbb{N}}$, so is an uncountable set.

The intervals $U_{n,\ell}$ which are deleted from $[0,1]$ to form $\mathbb{X}$ are called the "gaps" in the Cantor set $\mathbb{X}$. These gaps are in the complement of $\mathbb{X}$ in $\mathbb{R}$, and it is an exercise in the definition of $\mathbb{X}$ to see that each gap is an interval (a,b) where $a \in I_\sigma$ and $b \in I_\tau$ for $\sigma, \tau \in S_n$ for some n. These points a, b which are endpoints of gaps are all contained in $\mathbb{X}$. For each n there are only finitely many endpoints of gaps, so the endpoints of gaps form a countable set. These are the "obvious" points in $\mathbb{X}$. For example, when $n = 2$ the endpoints of gaps are the points at the end of the intervals

Since $\mathbb{X}$ is uncountable, there are many more points in $\mathbb{X}$!

However, there is no open, non-empty interval $(a, b) \subset \mathbb{X}$. If $\mathbb{X} \cap (a, b) \neq \emptyset$, then for all $n > 0$, $\mathbb{X}_n \cap (a, b) \neq \emptyset$ and hence there is an interval $I_\sigma \cap (a, b) \neq \emptyset$ for some $\sigma \in S_n$. We can then find some $m > n$ and $\tau \in S_m$ so the subinterval $I_\tau \subset I_\sigma$ with $I_\tau \subset (a, b)$. Then the gaps next to the endpoints of I_τ have non-empty intersection with (a, b). Thus, $(a, b) \neq \emptyset$ implies the intersection $(a, b) \cap (\mathbb{R} - \mathbb{X})$ non-empty.

The Cantor set $\mathbb{X}$ is not connected. Given any gap $U_{n,\ell}$ and point $z \in U_{n,\ell}$ then $\mathbb{X} \subset (-\infty, z) \cup (z, \infty)$. Moreover, since $\mathbb{X}$ contains no intervals, for any open set U if $U \cap \mathbb{X} \neq \emptyset$, then $U \cap \mathbb{X}$ is not connected. The Cantor set is said to "totally disconnected".

There are many variations on the above construction – what matters only is that at each stage, all of the closed intervals get "chopped up", maybe into two equal pieces like the middle third construction, or maybe into a randomly varying collection of closed subintervals. "In nature," which means in the study of dynamical systems, all of these other variations of the construction occur naturally.

CHAPTER 2

Applications to Calculus

1. How "big" is the set of the rational numbers?

Let $\mathbb{Q}$ denote the rational numbers, and choose an ordering $\mathbb{Q} \cong \{r_1, r_2, r_3, \dots\}$ where $r_n = \dfrac{p_n}{q_n}$ in reduced form. Actually, we can do the following with any countable subset of $\mathbb{R}$ – all we need is to order the elements in the set. For each $N > 0$, define the open set

$$V_N = \bigcup_{n=1}^{\infty} (r_n - 2^{-(N+n)}, r_n + 2^{-(N+n)})$$

V_N is open because it is a countable union of open intervals. The set V_N also contains the set $\mathbb{Q}$, that is, all of the rational numbers. The rationals are dense in $\mathbb{R}$ – they approximate every real number – so maybe the set V_N is just the entire line?

The surprising fact is that V_N is very small – smaller than 2^{1-N}. Each interval $(r_n - 2^{-(N+n)}, r_n + 2^{-(N+n)})$ has length $2^{-(N+n)} + 2^{-(N+n)} = 2 \cdot 2^{-(N+n)}$. The "total length" of V is at most the infinite sum of the geometric series

$$2 \cdot \left\{2^{-(N+1)} + 2^{-(N+2)} + 2^{-(N+3)} + \cdots + 2^{-(N+n)} + \cdots \right\} =$$

Pick $\epsilon > 0$, then choose a positive integer $N > 0$ so that $2^{-N} < \epsilon/2$. It follows that the "length" of V_N is $2 \cdot 2^{-N} < 2\epsilon/2 = \epsilon$. This is surprising, or counter-intuitive, that there is an open set smaller than $\epsilon > 0$ which contains every rational number in it, as the rational "are dense in $\mathbb{R}$". The key to thinking of this set V_N is that the intervals $(r_n - 2^{-(N+n)}, r_n + 2^{-(N+n)})$ are getting very small very fast, so it is no trouble to squeeze in more and more of them. The set of rational numbers is countable, so we get them all.

2. The measure of a set

Consider a set $X \subset \mathbb{R}$. One way to "measure" the size of X is to count it. This gives three possible answers – X is finite, X is countably infinite, or X is uncountably infinite. In the middle of the nineteenth century, Cantor proved that the set $\mathbb{R}$ is uncountably infinite, so this third possibility not only makes logical sense, but happens.

Cantor's work was motivated by the study of the sets of points $X \subset \mathbb{R}$ for which a Fourier series $F(x) = \sum_{n=0}^{\infty} \{a_n \cos nx + b_n \sin nx\}$ converges to the function $f(x)$ which gives rise to the coefficients. Such sets X may be $\emptyset$, finite, countably infinite, or even uncountably infinite. What they cannot be, is a set of positive Lebesgue measure.

The idea of Lebesgue measure was introduced around 1910. It is another way to "measure" the size of a set. All finite sets, and even all countably infinite sets, have Lebesgue measure zero. So a set with positive Lebesgue measure must be uncountable, thus this is a new way to "measure" the size of very large sets. There are sets with Lebesgue measure zero that are also uncountably infinite, so it is not just the number of points that matters, but how these points are placed in $\mathbb{R}$.

Definition 2.1 [Lebesgue measure zero sets]
We say that a subset $X \subset \mathbb{R}$ has *Lebesgue measure zero*, if for all $\epsilon > 0$, there exists a countable collection of open intervals $U_n = (a_n, b_n)$ so that $X \subset \bigcup_{n=1}^{\infty}(a_n, b_n)$ and $\sum_{n=1}^{\infty} |b_n - a_n| < \epsilon$.

The integers have Lebesgue measure zero. In the previous section, we proved that the rational numbers $\mathbb{Q}$ have Lebesgue measure 0.

The "middle third" Cantor set $\mathbb{X} \subset [0,1]$ constructed above has Lebesgue measure zero, even though it is bijective with the set $[0,1]$, so is uncountable. The proof that $\mathbb{X}$ has measure zero is easy - you show that $[0,1]-\mathbb{X}$ has measure 1. Since $[0,1]$ also has measure 1, then $1 + m(\mathbb{X}) = 1$ implies $m(\mathbb{X}) = 0$. Now, $[0,1] - \mathbb{X}$ has measure equal to the sum of the lengths of all the intervals removed, which is $1/3 + 2/9 + 4/27 + \cdots$. This is 1/3 times the geometric series for $r = 2/3$, so has sum $1/3 \cdot 1/(1 - 2/3) = 1$.

PROPOSITION 2.2. **[countable unions of measure zero sets]**
Let X_n be a set of Lebesgue measure zero for $n = 1, 2, 3, \ldots$. Then the union $X = \bigcup_{n=1}^{\infty} X_n$ has Lebesgue measure zero.

For example, a point has measure zero, so a countable unions of points (i.e., a countable set) has measure zero. But this proposition says more, that a countable union of anything with measure zero has measure zero.

We don't give the proof of the proposition, but sketch an outline. Given $\epsilon > 0$, choose an open cover for the set X_n whose length is at most $\epsilon/2^{-n}$. This exists as X_n has measure zero. The union of all these open sets, for all the X_n, is a countable set of countable sets, hence is countable. The union of the open sets then covers the union of the X_n, and have length at most ϵ. $\square$

The complete theory of Lebesgue measure, and especially of set with positive Lebesgue measure, takes much more care to define and develop. For an interval $[a,b]$, with or without the endpoints, its Lebesgue measure is $m([a,b]) = m([a,b)] = b-a$. When X is a countable union of disjoint open intervals - i.e. an open set in $\mathbb{R}$, the Lebesgue measure $m(X)$ is the sum of the lengths of its intervals.

Here is the very briefest definition of the Lebesgue measure $m(X)$ of $X \subset \mathbb{R}$. First, assume X is contained in a closed interval $[a,b]$. This can be accomplished by intersecting X with the intervals $[n,n+1]$ and adding the measure of each $X \cap [n,n+1]$ for all integers n. That is, we set $m(X) = \displaystyle\sum_{n=-\infty}^{\infty} m(X \cap [n,n+1])$ assuming that each term exists.

Now assume $X \subset [a,b]$. Then cover X with open sets and take the least upper bound of the length of such open covers. This is called the *outer measure* of X, and written $\overline{m}(X)$. Always, $0 \leq \overline{m}(X) \leq b-a$.

Invert the set and do the same for $[a,b]-X$. That is, $(b-a) - \underline{m}(X)$ is the least upper bound of the lengths of open covers of $[a,b]-X$. $\underline{m}(X)$ is called the *inner measure* of X. Always, $0 \leq \underline{m}(X) \leq b-a$.

You can show $\underline{m}(X) \leq \overline{m}(X)$ always holds. If they are equal, then the Lebesgue measure of X is $m(X) = \underline{m}(X) = \overline{m}(X)$.

There are sets $X \subset [0,1]$ for which the inner measure is less than the outer measure, $0 = \underline{m}(X) < \overline{m}(X) = 1$. Such a set X cannot be open or closed, and is constructed using the Axiom of Choice. These sets are always uncountable.

3. On the functions which are Riemann integrable

Recall that a function f is Riemann integrable on $[a,b]$ if for all $\epsilon > 0$, there exists a partition
$\mathcal{P} = \{a = t_0 < t_1 < \cdots t_n = b\}$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$. A basic result of Calculus is then

PROPOSITION 3.1. *f is continuous on $[a,b] \Longrightarrow f$ is Riemann integrable on $[a,b]$.*

Proof: f continuous implies f is uniformly continuous on $[a,b]$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|x-y| < \delta \Longrightarrow |f(x)-f(y)| < \dfrac{\epsilon}{2(b-a)}$. Choose $n > (b-a)/\delta$ then define the partition $\mathcal{P}$ with $t_i = a + i \cdot \Delta t$ where $\Delta t = (b-a)/n$. Then $\Delta t < \delta$ implies

$$M_i - m_i = \max\{f(x) \mid t_{i-1} \leq x \leq t_i\} - \min\{f(x) \mid t_{i-1} \leq x \leq t_i\} < \frac{\epsilon}{2(b-a)}$$

We obtain the estimate

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \sum_{i=1}^{n} (M_i - m_i)\Delta t \leq n \cdot \frac{\epsilon}{2(b-a)} \cdot \frac{(b-a)}{n} = \epsilon/2 < \epsilon \;\Box$$

But it is well known that the converse to this is false – if f is a function which is continuous at all but a finite number of points of $[a,b]$, then f is Riemann integrable on $[a,b]$.

Recall that C_f is the set of points where f is continuous, and then we set $D_f - \mathbb{R} - C_f$ which is the set of points where f is discontinuous. If $D_f \cap [a,b]$ is empty or finite, then f is Riemann integrable on $[a,b]$. The question is, how bad (large) can D_f be if f is Riemann integrable? This was answered by Lebesgue himself:

THEOREM 3.2. *A bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable $\Longleftrightarrow$ D_f has Lebesgue measure zero.*

Proof: The set D_f is the complement of $C_f = \bigcap_{n=1}^{\infty} U_n$, so if we set $D_n = (\mathbb{R} - U_n) \cap [a,b]$ then

$$D = D_f \cap [a,b] = \bigcup_{n=1}^{\infty} (\mathbb{R} - U_n) \cap [a,b] = \bigcup_{n=1}^{\infty} D_n$$

Assume f is integrable on $[a,b]$. We will show that each set D_n has measure zero. The countable union of measure zero sets has measure zero, so this will show that D has measure zero. Fix an integer $n > 0$. For any $\epsilon > 0$, we produce an open covering of D_n with measure less than ϵ. This will imply D_n has measure 0.

There is a partition $\mathcal{P}$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon/2n$. Let $a = t_0 < t_1 < \cdots t_k = b$ be the points of the partition. The collection of open intervals $\{J_i = (t_i - \epsilon/4(k+1), t_i + \epsilon/4(k+1)) \mid 0 \leq i \leq k\}$ which have total length $\epsilon/2$ and cover the points $\{t_0, t_1, \ldots, t_n\}$ of the partition $\mathcal{P}$, hence they also cover the points of $D_n \cap \mathcal{P}$.

On the other hand, let $I_1, I_2, \ldots, I_m$ be the open intervals of the partition $\mathcal{P}$ for which $(t_{i-1}, t_i) \cap D_n \neq \emptyset$. Given $z \in (t_{i-1}, t_i) \cap D_n$ if $|f(y) - f(x)| < 1/n$ for all $x, y \in (t_{i-1}, t_i)$ then $(t_{i-1}, t_i) \subset U_n$, which would contradict $z \in D_n$. So there must be some $x, y \in (t_{i-1}, t_i)$ with $|f(y) - f(x)| \geq 1/n$. This implies the difference $M_i - m_i \geq 1/n$, so $(t_i - t_{i-1})(M_i - m_i) \geq (t_i - t_{i-1})/n$ for each such interval. Hence

$$(|I_1| + |I_2| + \cdots + |I_m|)/n \leq U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon/2n$$

which implies $|I_1| + \cdots + |I_m| < \epsilon/2$. Thus the collection of intervals $\{J_0, \ldots, J_k, I_1, \ldots I_m\}$ covers D_m and has total length at most $\epsilon/2 + \epsilon/2 = \epsilon$.

Now assume that D has measure 0. Given $\epsilon > 0$, we show there exists a partition $\mathcal{P} = \{t_0 < \cdots < t_n\}$ of $[a,b]$ with $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$.

Let M be a bound for f of $[a,b]$. That is, $f(x) \leq M$ for all $a \leq x \leq b$.

Choose $n > 2(b-a)/\epsilon$ which implies $(b-a)/n < \epsilon/2$. The set $D_n \subset D$ so D_n has measure 0, hence there is a covering of D_n by open intervals $\{I_\alpha\}$ with length less than $\epsilon/4B$. Since U_n is open, $D_n = [a,b] \cap (\mathbb{R} - U_n)$ is a closed and bounded set. It follows that there is a finite subset $\{I_{\alpha_1}, I_{\alpha_2}, \ldots, I_{\alpha_k}\} \subset \{I_\alpha\}$ of the intervals which also covers D_n. The total length of the intervals $\{I_{\alpha_1}, I_{\alpha_2}, \ldots, I_{\alpha_k}\}$ is at most $\epsilon/4B$. Let I denote the open set $I = \bigcup_{i=1}^{k} I_{\alpha_i}$.

For all points $z \in [a,b]-I$, $z \notin I \Longrightarrow z \in G_n$ so there is an interval (a_z, b_z) containing z such that $x, y \in (a_z, b_z) \Longrightarrow |f(x) - f(y)| < 1/n$.

The collection of open intervals $\{(a_z, b_z) \mid z \in [a,b] - I\}$ covers the set $[a,b] - I$. This latter set is closed and bounded, hence is compact. So, there is a finite subcover by intervals $J_i, \ldots, J_m$. Let $\mathcal{P}$ be the partition formed from the endpoints of all the intervals J_i and I_{α_i}. Then

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) \le 2B\epsilon/4B + (b-a)/n < \epsilon/2 + \epsilon/2 = \epsilon$$

We are done. □

4. Baire category

There is yet another way to measure how big a set is, called its *Baire category.* The idea of the Baire category of a set may be the most subtile idea of these notes. It deserves a much more extensive discussion, but as the concept arises in the next section, we explain the definition here.

Consider a set $A \subset \mathbb{R}$. A point $x \in A$ is an *interior point* if there is some $\epsilon > 0$ so that $(x-\epsilon, x+\epsilon) \subset A$. For example, every point in an open set is an interior point – that is just the definition of an open set. The interior points of $[a,b]$ is the open interval (a,b).

We say a set A has *no interior* if the set of interior points is empty. In other words, this means that for any $x \in A$ and all $\epsilon > 0$, the interval $(x-\epsilon, x+\epsilon)$ intersects the complement of A. A finite set has no interior.

For the Cantor set $\mathbb{X}$, there are no interior points. To prove this is an exercise – the Cantor set is constructed by removing "middle thirds", and at stage n of the construction, the lengths of the closed intervals remaining are all $1/3^n$, so decreases to zero with n. If some interval $(a,b) \subset \mathbb{X}$ we get a contradiction, as the length of (a,b) is $b-a$ which does not go to zero with n.

A closed set with no interior points is said to *be of first category.* The general set of first category is a countable unions of such sets.

Definition 4.1 A set $X \subset \mathbb{R}$ is *of the first category* if it equals the countable union of closed sets in $\mathbb{R}$, $X = \bigcup_{n=1}^{\infty} X_n$ where each closed set X_n has no interior. If X is not of the first category, then X is said to be *of the second category.*

There is another way to view of sets of the Baire first category. Recall that a set $X \subset \mathbb{R}$ is dense means that for every open interval (a, b), the intersection $X \cap (a, b) \neq \emptyset$. Since every open set is the union of open intervals, this condition for intervals is equivalent to the condition that for every non-empty open set $U \subset \mathbb{R}$, the intersection $X \cap U \neq \emptyset$.

LEMMA 4.2. *A has no interior points $\Longleftrightarrow U = \mathbb{R} - A$ is dense in $\mathbb{R}$.*

Proof: Suppose A has no interior points. We show that $\mathbb{R} - A$ is dense. Suppose not, then there exists some non-empty open interval (a, b) with $(\mathbb{R} - A) \cap (a, b) = \emptyset$. But this implies the interval $(a, b) \subset A$, and then all of the points of (a, b) are interior for A. This is a contradiction, as A has no interior points.

Conversely, suppose that $\mathbb{R} - A$ is dense. Let $x \in A$ and $\epsilon > 0$. Then the open interval $(x - \epsilon, x + \epsilon)$ is non-empty, so $\mathbb{R} - A$ dense implies $(\mathbb{R} - A) \cap (x - \epsilon, x + \epsilon) \neq \emptyset$. But this means $(x - \epsilon, x + \epsilon) \not\subset A$ for all $\epsilon > 0$, which means x is not an interior point. This is true for every $x \in A$, so A has no interior points. □

While an open dense set has lots of open intervals everywhere, there are no intervals contained in its complementary set – none, anywhere.

A set with no interior points is said to be *nowhere dense.*

THEOREM 4.3. *Suppose that $\{U_n \mid n = 1, 2, 3, \ldots\}$ is a countable collection of open and dense subsets of $\mathbb{R}$. Then the intersection $\bigcap_{n=1}^{\infty} U_n$ is also dense in $\mathbb{R}$. In particular, it is not the empty set.*

COROLLARY 4.4. *Suppose that $\{A_n \mid n = 1, 2, 3, \ldots\}$ is a countable collection of closed and nowhere dense subsets of $\mathbb{R}$. Then the union $\bigcup_{n=1}^{\infty} A_n$ is nowhere dense in $\mathbb{R}$. In particular, it is not the entire set $\mathbb{R}$.*

In other words, the real line $\mathbb{R}$ is not the countable union of nowhere dense closed subsets. The set $\mathbb{R}$ is *of the second category*, and a countable union of nowhere dense closed sets is *of the first category.* A set of the first category is everywhere full of "holes" as its complement is dense in $\mathbb{R}$. This is why the Baire category is a type of "size" - a set of the first category can be huge, even dense, but it is never all of $\mathbb{R}$.

Proof: Assume $\{U_n \mid n = 1, 2, 3, \ldots\}$ is a countable collection of open and dense subsets of $\mathbb{R}$. Given a non-empty open set (a, b) we must show that $(a, b) \cap \bigcap_{n=1}^{\infty} U_n \neq \emptyset$.

We will use the Nested Interval Theorem 8.1 to find a point in this intersection. We construct the nested intervals $I_n = [a_n, b_n]$ using induction.

Chose a', b' with $a < a' < b' < b$. Since U_1 is dense there is an $x_1 \in (a', b') \cap U_1$. Since U_1 is open, there is an interval $x_1 \in (a'', b'') \subset U_1$. Let $a_1 = \max\{a', a''\}$ and $b_1 = \min\{b', b''\}$, so $a < a' \leq a_1 < b_1 \leq b' < b$. The open interval $(a_1, b_1) \subset U_1$, and we set $I_1 = [a_1, b_1]$.

Now assume for $1 \leq i \leq n$ we have points $a_i < b_i$ with $a_{i-1} < a_i < b_i < b_{i-1}$ and $(a_i, b_i) \subset U_i$. Choose $a' < b'$ with $a_n < a' < b' < b_n$. Since U_{n+1} is dense, there is an $x_{n+1} \in (a', b') \cap U_{n+1}$. Since U_{n+1} is open, there is an interval (a'', b'') with $x_{n+1} \in (a'', b'') \subset U_{n+1}$. Let $a_{n+1} = \max\{a', a''\}$ and $b_{n+1} = \min\{b', b''\}$. Then $a_n < a_{n+1} < b_{n+1} < b_n$ and $(a_{n+1}, b_{n+1}) \subset U_{n+1}$. Set $I_{n+1} = [a_{n+1}, b_{n+1}]$.

We have constructed nested closed intervals $I_1 \supset I_2 \supset \cdots$. By Theorem 8.1, or more precisely from its proof, the limits $\alpha = \sup\{a_n\}$ and $\beta = \inf\{b_n\}$ exist and satisfy $a < a_n \leq \alpha \leq \beta \leq b_n < b$ for all n.

Thus, the interval $[\alpha, \beta] \subset (a, b)$, and $[\alpha, \beta] \subset I_n \subset U_n$ for all n so $[\alpha, \beta] \subset \bigcap_{n=1}^{\infty} U_n$. It follows that $\emptyset \neq [\alpha, \beta] \subset (a, b) \cap \bigcap_{n=1}^{\infty} U_n$. $\Box$

We earlier constructed sets V_N whose Lebesgue measure is at most 2^{1-N} and $\mathbb{Q} \subset V_N$ for all N. The infinite intersection $V = \bigcap_{n=1}^{\infty} V_N$ contains $\mathbb{Q}$, but its measure is less than ϵ for all ϵ. So, the set V contains the rational numbers and has zero measure. It is natural to ask, does $V = \mathbb{Q}$, or are there extra points in it beside $\mathbb{Q}$? The following result says there are lots of extra points.

Proposition 4.5. *The set of rational numbers is not the intersection of a countable collection of open dense subsets of* $\mathbb{R}$.

Proof: Let $\{U_n\}$ be a countable collection of open dense subsets of $\mathbb{R}$. Suppose that $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$ – we show this leads to a contradiction.

Choose an ordering $\mathbb{Q} \cong \{r_1, r_2, r_3, \ldots\}$ in reduced form. For each n, define the open and dense set $W_n = \mathbb{R} - \{r_n\}$. Now, combine the two countable collections of open and dense subsets, to get countable collection of open and dense subsets $\{U_1, U_2, \ldots, W_1, W_2, \ldots\}$. By the Baire Theorem 4.3, the intersection

$$\bigcap_{n=1}^{\infty} U_n \cap \bigcap_{m=1}^{\infty} W_m$$

is dense in $\mathbb{R}$. But the first intersection contains only the set $\mathbb{Q}$, while the latter intersection explicitly contains none of the rationals. Thus, the intersection is empty, which is a contradiction. $\square$

The proof actually shows that any countable subset $X \subset \mathbb{R}$ is not the countable intersection of open and dense subsets. There was nothing special about the rationals, other than they were countable.

5. On the set of points where a function is continuous

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function, and let

$$C_f = \{x \mid f \text{ is continuous at } x\}$$

For example, the Dirichlet function $f(x) = \begin{cases} 1/q & \text{for} \quad x = p/q \\ 0 & \text{for} \quad x \text{ irrational} \end{cases}$ is continuous exactly on the set $C_f = (\mathbb{R} - \mathbb{Q}) \cup \{0\}$, which is the set of irrational numbers plus $\{0\}$.

We can also construct a function with $C_f = \mathbb{R} - \mathbb{Q}$ – choose an ordering $\mathbb{Q} \cong \{r_1, r_2, r_3, \dots\}$, then set $f(x) = \begin{cases} 1/n & \text{for} \quad x = r_n \\ 0 & \text{for} \quad x \text{ irrational} \end{cases}$

In both these cases, the set of points of discontinuity, the complement $\mathbb{R} - C_f$, is countable.

The general question is, what sets can occur as C_f for some f? Does $\mathbb{Q} = C_f$ for some f? Is it possible to construct a function which is *continuous* exactly on the rational numbers?

Given a function f, set

$$U_n = \bigcup \{(a, b) \mid x, y \in (a, b) \Longrightarrow |f(x) - f(y)| < 1/n\}$$

So, U_n is the union of all open intervals (a, b) so that the image $f(a, b)$ has size less than $1/n$. As U_n is the union of open intervals, it is also an open set.

PROPOSITION 5.1. $C_f = \bigcap_{n=1}^{\infty} U_n$. *Thus, C_f is a countable intersection of open sets in $\mathbb{R}$.*

Proof: Given $z \in \bigcap_{n=1}^{\infty} U_n$ we show that f is continuous at z.

Given $\epsilon > 0$, choose an integer $n > 1/\epsilon$ so $1/n < \epsilon$.

The point $z \in U_n$ so there is some interval (a, b) with $z \in (a, b) \subset U_n$ hence $x, y \in (a, b) \Longrightarrow |f(x) - f(y)| < 1/n < \epsilon$.

Let $\delta = \min\{z-a, b-z\}$, so $(z-\delta, z+\delta) \subset (a,b)$. Then

$$|x-y| < \delta \Longrightarrow x, y \in (a,b) \Longrightarrow |f(x) - f(y)| < 1/n < \epsilon$$

This shows f is continuous at z hence $z \in C_f$.

Conversely, suppose $z \in C_f$. Given any integer $n > 0$, by the continuity of f at z, there exists $\delta > 0$ such that

$$|x-z| < \delta \Longrightarrow |f(x) - f(y)| < 1/2n$$

So we take $a = z - \delta$ and $b = z + \delta$, then

$$\begin{aligned} x, y \in (a,b) &\Longrightarrow |x-z| < \delta \ \&\ |y-z| < \delta \\ &\Longrightarrow |f(x) - f(z)| < 1/2n \ \&\ |f(y) - f(z)| < 1/2n \\ &\Longrightarrow |f(x) - f(y)| < 1/n \end{aligned}$$

Therefore, $z \in (a,b) \subset U_n$. This holds for all n, so $z \in \bigcap_{n=1}^{\infty} U_n$. $\square$

This allows us to answer the question: is there a function for which $\mathbb{Q} = C_f$? If there is such a function, then $\mathbb{Q} = C_f = \bigcap_{n=1}^{\infty} U_n$ so $\mathbb{Q} \subset U_n$ for all n. This means that each set U_n is not only open, but also dense. But we showed that $\mathbb{Q}$ is not the intersection of a countable collection of open dense subsets, so this is a contradiction. We have shown:

Corollary 5.2. *There is no function f with $C_f = \mathbb{Q}$.*

There is a modified version of this question: Given a countable collection of open sets $\{V_n\}$, is there f with $C_f = \cap V_n$?

The answer is yes, and the surprising fact is that the proof is almost the same as what we did to construct the function continuous on the irrational numbers.

Proposition 5.3. *Let $V_n \subset \mathbb{R}$ be an open set for each $n = 1, 2, 3, \ldots$. Then there is a function $f \colon \mathbb{R} \to \mathbb{R}$ such that $C_f = \bigcap_{n=1}^{\infty} V_n$.*

Proof: The open sets V_n are not assumed to be nested, so we first define a sequence of nested open sets whose infinite intersection is $C = \bigcap_{n=1}^{\infty} V_n$. Set $U_1 = \mathbb{R}$. Set $U_2 = V_1 \cap V_2$. For $n > 2$ set $U_n = V_n \cap U_{n-1}$.

Then each U_n is open, and they are nested: $U_1 \supset U_2 \supset \cdots$.

Moreover, $C = \bigcap_{n=1}^{\infty} U_n$.

Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in C \\ 1/n & \text{if} \quad x \in U_n, x \notin U_{n+1}, x \in \mathbb{Q} \\ -1/n & \text{if} \quad x \in U_n, x \notin U_{n+1}, x \notin \mathbb{Q} \end{cases}$$

To show $C = C_f$ we first show that $C \subset C_f$. That is, if $x \in U_n$ for every n then f is continuous at x.

Fix $x \in C$, and let $\epsilon > 0$. Choose $n > 1/\epsilon$ so $1/n < \epsilon$. Since $x \in U_n$ and U_n is open, there is a $\delta > 0$ such that $(x - \delta, x + \delta) \subset U_n$. Then

$$|y - x| < \delta \Longrightarrow y \in U_n \Longrightarrow |f(y) - f(x)| = |f(y) - 0| = |f(y)| \leq 1/n < \epsilon$$

Next, we show that if $x \notin C$ then f is not continuous at x, or $x \notin C_f$.

Fix $x \notin C$. Then $U_1 = \mathbb{R}$, so $x \in U_n$ for some n. Let n be the greatest integer such that $x \in U_n$ but $x \notin U_{n+1}$. Then $f(x) = \pm 1/n$. If $x \in \mathbb{Q}$ then $f(x) = 1/n$, otherwise $f(x) = -1/n$.

Given $\delta > 0$, suppose that $x \in \mathbb{Q}$ then there is some irrational number $y \in (x - \delta, x + \delta) \cap U_n$ as the intersection is an open set and the irrationals are dense. Then $f(y) \leq 0$, so $|f(y) - f(x)| \geq 1/n$. This means that if we take $\epsilon < 1/n$ then for all $\delta > 0$, there is some $y \in (x - \delta, x + \delta)$ such that $|f(y) - f(x)| \not< \epsilon$ so f is not continuous at x.

Given $\delta > 0$, suppose that $x \notin \mathbb{Q}$ then there is some rational number $y \in (x - \delta, x + \delta) \cap U_n$ as the intersection is an open set and the rationals are dense. Then $f(y) \geq 0$, so $|f(y) - f(x)| \geq 1/n$. This means that if we take $\epsilon < 1/n$ then for all $\delta > 0$, there is some $y \in (x - \delta, x + \delta)$ such that $|f(y) - f(x)| \not< \epsilon$ so f again is not continuous at x. □

For example, choose an ordering $\mathbb{Q} \cong \{r_1, r_2, r_3, \dots\}$.

Set $V_n = \mathbb{R} - \{r_n\}$ and apply the above construction.

The resulting f is exactly the function at the beginning of this section which is continuous precisely at the irrational numbers.

6. Uniform convergence of functions

Suppose there is given a subset $X \subset \mathbb{R}$ and a sequence of functions $\{f_n\colon X \to \mathbb{R} \mid n = 1, 2, \dots\}$ and there is a function $f\colon X \to \mathbb{R}$ such that for each $x \in X$, the limit exist $\lim\limits_{n\to\infty} f_n(x) = f(x)$. The question is, what properties does $f(x)$ have?

- If each f_n is continuous, must f be continuous?
- If each f_n is Riemann integrable, must f be Riemann integrable?
- If each f_n is differentiable, must f be differentiable?

The answer is NO in general – but in the first two cases, it is yes if we demand that the convergence is uniform.

Definition 6.1 Let $X \subset \mathbb{R}$ and $f_n \colon X \to \mathbb{R}$, $n = 1, 2, \ldots$ be a sequence of functions. We say that $\{f_n\}$ converges uniformly on X to $f \colon X \to \mathbb{R}$ if

$$\forall \epsilon > 0, \exists N > 0 \text{ such that } n > N \Longrightarrow \forall x \in X, |f_n(x) - f(x)| < \epsilon$$

The point of the definition is that given $\epsilon > 0$, the choice of N works for all $x \in X$. If we put the choice of x first,

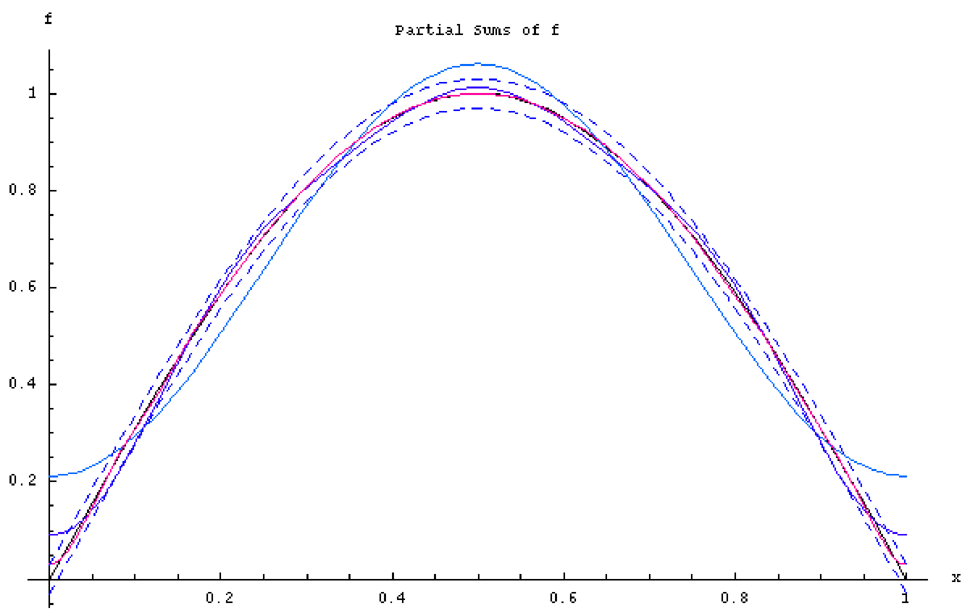
$$\forall x \in X, \forall \epsilon > 0, \exists N > 0 \text{ such that } n > N \Longrightarrow |f_n(x) - f(x)| < \epsilon$$

Here, N depends on both x and ϵ, and this is just the definition that $\forall x \in X, f_n(x) \to f(x)$.

Another way to intuitively understand uniform convergence, is to look at the graph of the limit function $y = f(x)$. Given $\epsilon > 0$, form the strip (or tube) around the graph, consisting of all points

$$Tube(f, \epsilon) = \{(x, y) \mid a \leq x \leq b \;\&\; |y - f(x)| < \epsilon\}$$

The requirement is that for any $\epsilon > 0$ there is some N so that all of the functions f_n for $n > N$ have graphs contained in this tube. Here is a sample illustration of this:



We prove three results, corresponding to the three questions above. The first result is that a uniformly continuous limit of continuous functions is continuous. We state this for the restriction of the functions to some subset $X \subset \mathbb{R}$ – though in most cases, $X = [a, b]$.

THEOREM 6.2. *Let $X \subset \mathbb{R}$ and suppose $\{f_n \colon X \to \mathbb{R} \mid n = 1, 2, \ldots\}$ is a sequence of continuous functions which converges uniformly on X to $f \colon X \to \mathbb{R}$. Then f is continuous on X.*

Proof: For $z \in X$ and $\epsilon > 0$, we need to find $\delta > 0$ so that $\forall y \in X, |z-y| < \delta \Longrightarrow |f(z) - f(y)| < \epsilon$.

By the definition of uniform convergence, for $\epsilon/3$ there exists $N > 0$ such that

$$n > N \Longrightarrow \forall x \in X, |f_n(x) - f(x)| < \epsilon/3$$

Fix $n > N$, then f_n is continuous on X by assumption, so for $z \in X$, there exists $\delta > 0$ such that $\forall y \in X, |z-y| < \delta \Longrightarrow |f_n(z) - f_n(y)| < \epsilon$. Then $\forall y \in X, |z-y| < \delta$ implies

$$\begin{aligned} |f(z) - f(y)| &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(y)| + |f_n(y) - f(y)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

Thus, $\forall y \in X, |z-y| < \delta \Longrightarrow |f(z) - f(y)|$. □

The second result is that a uniformly continuous limit of integrable functions is integrable.

THEOREM 6.3. *Let $\{f_n \colon [a,b] \to \mathbb{R} \mid n = 1, 2, \dots\}$ be a sequence of functions which converges uniformly on $[a,b]$ to $f \colon [a,b] \to \mathbb{R}$. If each f_n is Riemann integrable on $[a,b]$, then f is Riemann integrable on $[a,b]$, and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.*

Proof: We will show that for any $\epsilon > 0$, there is a partition $\mathcal{P}$ of $[a,b]$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.

Fix $\epsilon > 0$, and choose $N > 0$ so that

$$n > N \Longrightarrow \forall x \in [a,b],\ |f_n(x) - f(x)| < \frac{\epsilon}{4(b-a)}$$

Fix $n > N$, then f_n is Riemann integrable, so there is a partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a,b]$ such that

$$U(f_n, \mathcal{P}) - L(f_n, \mathcal{P}) < \epsilon/2$$

For each $t_{i-1} \leq x \leq t_i$ we have $|f_n(x) - f(x)| < \epsilon/4(b-a)$ so

$$\begin{aligned} M_i &= \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\} &\leq \sup\{f_n(x) \mid t_{i-1} \leq x \leq t_i\} &+ \epsilon/4(b-a) \\ m_i &= \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} &\geq \inf\{f_n(x) \mid t_{i-1} \leq x \leq t_i\} &- \epsilon/4(b-a) \end{aligned}$$

Then calculate

$$
\begin{aligned}
&U(f,\mathcal{P}) - L(f,\mathcal{P}) \\
&= \sum_{i=1}^{m} (M_i - n_i)(t_i - t_{i-1}) \\
&< \sum_{i=1}^{m} \left(\sup\{f_n(x) \mid t_{i-1} \le x \le t_i\} - \inf\{f_n(x) \mid t_{i-1} \le x \le t_i\} + 2\epsilon/4(b-a)\right)(t_i - t_{i-1}) \\
&\le \sum_{i=1}^{m} \left(\sup\{f_n(x) \mid t_{i-1} \le x \le t_i\} - \inf\{f_n(x) \mid t_{i-1} \le x \le t_i\}\right)(t_i - t_{i-1}) + \epsilon/2 \\
&= U(f_n,\mathcal{P}) - L(f_n,\mathcal{P}) + \epsilon/2 = \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, this shows f is Riemann integrable.

For this ϵ and N, we also have $|U(f,\mathcal{P}) - U(f_n,\mathcal{P})| < \epsilon/2$ and hence

$$\left|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\right| \le |U(f,\mathcal{P}) - U(f_n,\mathcal{P})| + \epsilon + \epsilon/2 < 2\epsilon$$

Thus, the limit of the integrals is the integral of the limit. □

The third result gives a condition which guarantees that a uniformly continuous limit of smooth functions is smooth. One version of this result assumes the derivative functions f_n' are integrable on $[a,b]$, then the proof follows easily from the Fundamental Theorem of Calculus and Theorem 6.3 above. For example, this proof is given in Chapter 24 of Spivak's book "Calculus". The theorem below assumes only that the derivatives f_n' exists at every point. The proof is then more direct and delicate, and uses the Mean Value Theorem. (The following proof is from the notes "Analysis From Scratch" by Peter Kropholler.)

THEOREM 6.4. *Let $\{f_n \colon [a,b] \to \mathbb{R} \mid n = 1,2,\ldots\}$ be a sequence of continuously differentiable functions on (a,b). Suppose that both the sequences $\{f_n\}$ and $\{f_n'\}$ converge uniformly to functions f and g. Then f is differentiable, and for each $x \in (a,b)$, $f'(x) = g(x)$. In other words, the derivative of the limit equals the limit of the derivatives.*

Proof: Fix $x \in (a,b)$ We must show $f'(x) = g(x)$, or that

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = g(x)$$

Let $\epsilon > 0$ be given, then we must show there is a $\delta > 0$ such that

$$|y - x| < \delta \Longrightarrow \left|\frac{f(y) - f(x)}{y - x} - g(x)\right| < \epsilon$$

Choose $N > 0$ so that $n > N \Longrightarrow \forall \xi \in (a,b),\ |f_n'(\xi) - g(\xi)| < \epsilon/4$.

The functions f_n' are continuous, and converge uniformly to g, so by Theorem 6.2, g is continuous on (a,b). Choose $\delta > 0$ so that

$$|y - x| < \delta \Longrightarrow |g(y) - g(x)| < \epsilon/4$$

Fix $n > N$. Then for any $y \neq x$, the Mean Value Theorem implies there exists ξ between x and y such that

$$\frac{f_n(y) - f_n(x)}{y - x} = f_n'(\xi)$$

Now, put all this together, for $n > N$ and $0 < |y - x| < \delta$,

$$\begin{aligned} \left| \frac{f_n(y) - f_n(x)}{y - x} - g(x) \right| &= |f_n'(\xi) - g(x)| \\ &= |f_n'(\xi) - g(\xi) + g(\xi) - g(x)| \\ &\leq |f_n'(\xi) - g(\xi)| + |g(\xi) - g(x)| \\ &\leq \epsilon/4 + \epsilon/4 = \epsilon/2 \end{aligned}$$

We must now replace the functions f_n with the limit f in this estimate, and we are done. Fix the point y satisfying $0 < |y - x| < \delta$, then $\epsilon|y - x|/4 > 0$ is a positive constant. Then by the uniform convergence of $f_n \to f$ we can choose $n > N$ large enough so that both

$$\begin{aligned} |f(x) - f_n(x)| &< \epsilon|y - x|/4 \\ |f(y) - f_n(y)| &< \epsilon|y - x|/4 \end{aligned}$$

Then

$$\left| \frac{f(y) - f(x)}{y - x} - \frac{f_n(y) - f_n(x)}{y - x} \right| < \epsilon/4 + \epsilon/4 = \epsilon/2$$

so finally,

$$\begin{aligned} &\left| \frac{f(y) - f(x)}{y - x} - g(x) \right| \\ &\leq \left| \frac{f(y) - f(x)}{y - x} - \frac{f_n(y) - f_n(x)}{y - x} \right| + \left| \frac{f_n(y) - f_n(x)}{y - x} - g(x) \right| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

□

7. The Cantor Function

Recall the Cantor set is described as the intersection $\mathbb{X} = \bigcap_{n>0} \mathbb{X}_n$ where $\mathbb{X}_n = \bigcup_{\sigma \in S_n} I_\sigma$ is the union of the closed intervals I_σ defined by sequences $\sigma \in S_n$. The set $\mathbb{X}_n$ is the union of 2^n closed intervals, and its complement U_n in $[0,1]$ is the union of open intervals

$$V_k = V_{k,1} \cup \cdots \cup V_{k,2^{k-1}} \ , \ k = 1, \ldots, n$$

Then $\mathbb{X}$ is the complement in $[0,1]$ of the open set $U = \bigcup_{n>0} U_n$ which is the union of all the open intervals $U_{n,\ell}$, $n > 0$ and $1 \leq \ell < 2^n$. The Cantor function, or the "Devil's Staircase", is a remarkable function $\Theta \colon [0,1] \to [0,1]$ which is continuous, onto and constant on the set U. So, Θ has a derivative on the open set U, and $\Theta' = 0$ there. Thus $\Theta'(x) = 0$ for x in a set of Lebesgue measure 1, yet Θ is not constant. Here is its graph:

We will give three constructions of this remarkable function Θ.

Construction 7.1 For each $n > 0$ and $1 \leq \ell < 2^n$, set $\Theta(x) = \dfrac{2\ell - 1}{2^n}$ when $x \in V_{n,\ell}$. For example, $\Theta(x) = 1/2$ for $x \in V_{1,1}$ – the first middle third $(1/3, 2/3)$. Then $\Theta(x) = 1/4$ for $x \in V_{2,1}$ – the first middle ninth $(1/9, 2/9)$ – and $\Theta(x) = 3/4$ for $x \in V_{2,2}$ – the second middle ninth $(7/9, 8/9)$. In general, on the open sets $V_n = V_{n,1} \cup \cdots \cup V_{n,2^{n-1}}$, the values are

$$\frac{1}{2^n}, \frac{3}{2^n}, \frac{5}{2^n}, \ldots, \frac{2^n - 1}{2^n}$$

Note there are 2^{n-1} odd numbers less than 2^n and 2^{n-1} open intervals in V_n. The reason why we use odd numerators defining Θ on V_n, is that the values $p/2^n$ for the even numerator $p = 2^i\ell$, $i > 0$, occur for V_k where $k = n - i < n$.

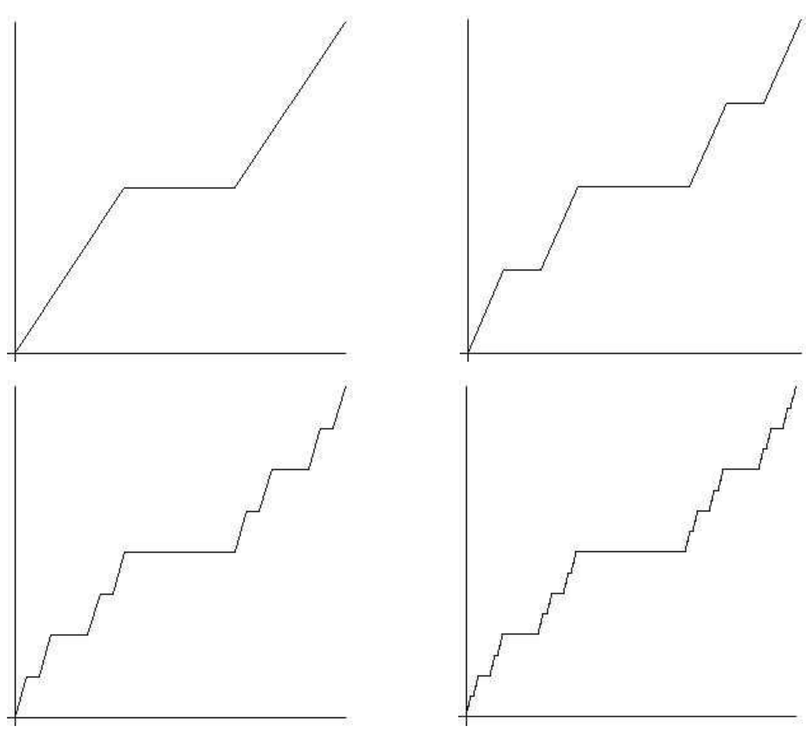
This defines Θ on the set U. Note that Θ is non-decreasing, and constant on open intervals of U. Now extend Θ to all of $[0,1]$ by setting $\Theta(0) = 0$, $\Theta(1) = 1$, and $\Theta(x) = \sup\{\Theta(z) \mid z \leq x \ \& \ z \in U\}$.

Then Θ is clearly non-decreasing on $[0,1]$, so it remains to show Θ is continuous. But this follows since the values of Θ on U is the set of all dyadic rationals in $[0,1]$ – the numbers of the form $p/2^n$. These are dense in $[0,1]$. Since Θ is non-decreasing, it is an exercise to show Θ not continuous at $x \in [0,1]$ implies there is a jump in the range of Θ - i.e.,

$$\lim_{h\to 0+} \Theta(x+h) - \lim_{h\to 0-} \Theta(x+h) = c > 0$$

and so the range of Θ omits an interval of length c. But the image is dense, so omits no interval. □

Construction 7.2 The second construction of the Cantor function Θ modifies the above, in that we produce a sequence of continuous functions $\theta_n \colon [0,1] \to [0,1]$ which converge uniformly to Θ. Here is a sequence of the graphs of the first four functions, θ_1, θ_2, θ_3, and θ_4:



Let $\Theta_n(0) = 0$ and $\Theta_n(1) = 1$ for all $n > 0$.

Define $\theta_1\colon [0,1] \to [0,1]$ to be $1/2$ on $U_1 = V_{1,1} = (1/3, 2/3)$ the middle third interval, and to be linear on the complement $[0,1] - U_1$. That is,

$$\theta_1(x) = \begin{cases} 3x/2 & \text{for} \quad 0 \le x \le 1/3 \\ 1/2 & \text{for} \quad x \in U_n, x \notin U_{n+1}, 1/3 \le x \le 2/3 \\ (3x-1)/2 & \text{for} \quad x \in U_n, x \notin U_{n+1}, 2/3 \le x \le 1 \end{cases}$$

In general, define θ_n on U_n by specifying that on the open sets $V_k = V_{k,1} \cup \cdots \cup V_{k,2^{k-1}}$ for $1 \le k \le n$, the values of θ_n are

$$\frac{1}{2^k}, \frac{3}{2^k}, \frac{5}{2^k}, \ldots, \frac{2^k-1}{2^k}$$

Thus, $\theta_n = \Theta$ on U_n. Extend θ_n to the complement $[0,1] - U_n$ by requiring that it be continuous and linear.

The sequence $\{\theta_n\}$ is uniformly Cauchy on $[0,1]$. For $x \in U_n$ when $m, m' > n$ we have $\theta_m(x) = \theta_{m'}(x) = \Theta(x)$ is unchanging with m. For $x \in \mathbb{X}_n = [0,1] - U_n$ and $m > n$, the construction yields $|\theta_m(x) - \theta_n(x)| \le 2^{-n}$. Thus, $\theta_n \to \Theta$ uniformly on $[0,1]$.

Construction 7.3 The third construction is a modification of the above, but now the functions θ_n are defined as the iterations of a given function. This idea is from the paper "The Standard Cantor function is subadditive" by Josef Doboŝ, in Proceedings of the American Math. Soc., vol. 124, 1996, pages 3425–3426.

Define the sequence of functions $\phi_n\colon \mathbb{R} \to [0,1]$ by

$$\begin{aligned} \phi_0(x) &= \begin{cases} 0 & \text{if} \quad x \le 0 \\ x & \text{if} \quad 0 \le x \le 1 \\ 1 & \text{if} \quad x \ge 1 \end{cases} \\ \phi_{n+1}(x) &= \begin{cases} \frac{1}{2} \cdot \phi_n(x) & \text{if} \quad x \le 2/3 \\ \frac{1}{2} + \frac{1}{2} \cdot \phi_n(3x-2) & \text{if} \quad x \ge 1/3 \end{cases} \end{aligned}$$

It is then easy to check that $\phi_n(0) = 0$ for all $x \le 0$, and $\phi_n(1) = 1$ for all $x \ge 1$. Moreover, the overlapping definitions of ϕ_{n+1} agree for $1/3 < x < 2/3$.

Then check that $\phi_n \to \Theta$ uniformly on $[0,1]$.

The details to check are all left as exercises.

8. Existence of solutions to differential equations

Suppose $f(x,y)$ is continuous in x and y and that $\frac{\partial f}{\partial y}$ exists and is continuous. Consider the differential equation $y' = f(x,y)$. We say the function $g(x)$ is a solution if $g'(x) = f(x, g(x))$. If g satisfies the initial condition $g(a) = b$ then, by the fundamental theorem of calculus, it is equivalent that g be continuous and satisfy the integral equation

$$g(x) = b + \int_a^x f(t, g(t))\,dt. \tag{1}$$

For a function g which is continuous near a, define a mapping T which takes g to a new function Tg defined by

$$(Tg)(x) = b + \int_a^x f(t, g(t))\,dt.$$

Then g is a solution to (1) if and only if $Tg = g$. The idea is to use T to construct a sequence of functions which converges uniformly to a solution to (1). The proof of convergence uses estimates which depend on bounds on f and $\frac{\partial f}{\partial y}$.

Theorem 8.1. *Assume $f(x,y)$ is continuous, $|f(x,y)| \le B$, and $|\frac{\partial f}{\partial y}(x,y)| \le M$ for points (x,y) in some rectangle R containing the point (a,b). Choose $\delta > 0$ such that*

(i) $\delta M \le r < 1$
(ii) If $|x-a| \le \delta$ and $|y-b| \le \delta B$, then (x,y) lies in R.

Then there is a unique function g defined for x with $|x-a| \le \delta$ such that $g(a) = b$ and $g'(x) = f(x, g(x))$.

Proof: Let $\mathcal{F}$ be the set of continuous functions g such that $g(a) = b$ and, if $|x-a| \le \delta$, then $|g(x) - b| \le \delta B$. Notice that the function $g_0(x) = b$ is in $\mathcal{F}$. Also notice that if g lies in $\mathcal{F}$ and $|x-a| \le \delta$, then

$$|(Tg)(x) - b| \le \int_a^x |f(t, g(t))|\,dt \le |x-a| \le \delta B$$

so Tg lies in $\mathcal{F}$. We want to prove that the sequence g_n defined inductively by

$$g_0(x) = b \text{ and } g_n = Tg_{n-1}$$

converges uniformly.

For a function g in $\mathcal{F}$, or generally for a bounded function g, define a norm

$$||g|| = \sup\{|g(x)| : |x-a| \le \delta\}.$$

If $||g|| = 0$, then $g(x) = 0$ for $|x-a| \le \delta$.

If g and h are functions in $\mathcal{F}$, then

1. $||g - h|| = ||h - g||$,
2. $||g - h|| = 0$ if and only if $g = h$,
3. $||h_1 - h_3|| \leq ||h_1 - h_2|| + ||h_2 - h_3||$.

The third property is called the triangle inequality by analogy with the formula for distances between points in the plane. To prove it notice that

$$|h_1(x) - h_3(x)| \leq |h_1(x) - h_2(x)| + |h_2(x) - h_3(x)| \leq ||h_1 - h_2|| + ||h_2 - h_3||$$

for any x with $|x - a| \leq \delta$, so the left hand side is bounded by the right hand side. These three properties make $\mathcal{F}$ a metric space with the distance between g and h given by $||g - h||$.

Now, we return to the proof of the Theorem. If (x, y_1) and (x, y_2) are in R then, by the mean value theorem,

$$|f(x, y_2) - f(x, y_1)| = \left|\frac{\partial f}{\partial y}(x, \eta)(y_2 - y_1)\right| \leq M|y_2 - y_1|. \tag{2}$$

Hence, if g and h are in $\mathcal{F}$ and $|x - a| \leq \delta$, then

$$\begin{aligned} |(Tg)(x) - (Th)(x)| &\leq \int_a^x |f(t, g(t)) - f(t, h(t))| \, dt \\ &\leq M \int_a^x |g(t) - h(t)| \, dt \\ &\leq M\delta ||g - h|| \end{aligned}$$

Thus $||Tg - Th|| \leq r||g - h||$. Since $r < 1$, T is called a contraction map; under T points (functions) are move closer together.

For the functions g_n in our sequence this implies $||g_2 - g_1|| \leq r||g_1 - g_0||$ and inductively

$$||g_n - g_{n-1}|| \leq r^{n-1}||g_1 - g_0||.$$

Therefore, with $m < n$,

$$\begin{aligned} ||g_n - g_m|| &\leq ||g_n - g_{n-1}|| + \cdots + ||g_{m+1} - g_m|| \\ &\leq (r^{n-1} + \cdots + r^m)||g_1 - g_0|| \\ &\leq (r^m + r^{m+1} + \cdots)||g_1 - g_0|| \\ &\leq \frac{r^m}{1 - r}||g_1 - g_0|| \end{aligned}$$

Since $r^m \to 0$ as $m \to \infty$, this shows the sequence g_n is uniformly Cauchy and hence converges uniformly to a continuous function g which, we can check, lies in $\mathcal{F}$.

To show that g is a solution to (1) we need to show that g is a fixed point of T. which follows from the fact that it is a contraction map. Without using metric space theory we may argue as follows. Since g_n converges uniformly to g, it follows from (2) that $f(t, g_n(t))$ converges uniformly to $f(t, g(t))$. Then

$$\begin{aligned}(Tg)(x) &= b + \int_a^x \lim_{n\to\infty} f(t, g_n(t))\,dt \\ &= \lim_{n\to\infty} \{b + \int_a^x f(t, g_n(t))\,dt\} \\ &= \lim_{n\to\infty} g_{n+1}(x) = g(x)\end{aligned}$$

Hence g is a solution to our differential equation.

If h were another solution, then $Th = h$. But

$$||g - h|| = ||Tg - Th|| \leq r||g - h||$$

and $r < 1$ imply $||g - h|| = 0$ and hence $g = h$. $\square$

9. Implicit Function Theorem

The contraction mapping technique can also be used to prove the implicit function theorem. The equation $f(x, y) = 0$ is said to define the function $g(x)$ implicitly if $f(x, g(x)) = 0$. If $f(a, b) = 0$ we may require $g(a) = b$.

THEOREM 9.1. *Suppose $f(a, b) = 0$, f is continuous near (a, b), $\frac{\partial f}{\partial y}(x, y)$ is continuous at (a, b), and $\frac{\partial f}{\partial y}(a, b) \neq 0$. Then there is a unique continuous function g defined for x near a with $g(a) = b$ and $f(x, g(x)) = 0$. Further, if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous near (a, b), then g is differentiable and*

$$g'(x) = -\frac{\partial f}{\partial x}(x, g(x)) \Big/ \frac{\partial f}{\partial y}(x, g(x)).$$

Proof: Let

$$F(x, y) = y - \left(\frac{\partial f}{\partial y}(a, b)\right)^{-1} f(x, y).$$

Then $F(a, b) = b$ and $\frac{\partial F}{\partial y}(a, b) = 0$, so $\frac{\partial F}{\partial y}(x, y)$ is small near (a, b).

Choose $\delta > 0$ and $k > 0$ so that if $|x - a| \le \delta$ and $|y - b| \le k$, then $|\frac{\partial f}{\partial y}(x, y)| \le \frac{1}{2}$ and, taking δ smaller if necessary, $|F(x, b) - b| \le \frac{k}{2}$.

Let $\mathcal{F}$ be the set of continuous functions g such that $g(a) = b$ and, if $|x - a| \le \delta$, then $|g(x) - b| \le k$.

For $g \in \mathcal{F}$ and $|x - a| \le \delta$, define $(Tg)(x) = F(x, g(x))$. Then $(Tg)(a) = b$ and

$$\begin{aligned}
|(Tg)(x) - b| &= |F(x, g(x)) - b| \\
&\le |F(x, g(x)) - F(x, b)| + |F(x, b) - b| \\
&\le |\frac{\partial F}{\partial y}(x, \eta)|\,|g(x) - b| + \frac{k}{2} \\
&\le \frac{k}{2} + \frac{k}{2} = k
\end{aligned}$$

Hence $Tg \in \mathcal{F}$.

T is a contraction mapping since

$$\begin{aligned}
&(1) \qquad |(Tg)(x) - (Th)(x)| = |F(x, g(x)) - F(x, h(x))| \\
&(2) \qquad\qquad = \left|\frac{\partial F}{\partial y}(x, \eta)\right| |g(x) - h(x)| \le \frac{1}{2}|g(x) - h(x)|
\end{aligned}$$

So

$$||Tg - Th|| \le \frac{1}{2}||g - h||$$

Set $g_0(x) = b$ and $g_{n+1} = Tg_n$. As before the sequence g_n is uniformly Cauchy and hence converges uniformly to a continuous function g. It follows from (3) that $F(x, g_n(x))$ converges uniformly to $F(x, g(x))$. Then

$$F(x, g(x)) = \lim_{n\to\infty} F(x, g_n(x)) = \lim_{n\to\infty} g_{n+1}(x) = g(x).$$

Therefore g is a fixed point of T. This means $F(x, g(x)) = g(x)$ and consequently $f(x, g(x)) = 0$.

Now assume $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(x, g(x))$ and let x_1 be close to x with $|x_1 - a| < \delta$.

Then $f(x, g(x)) = 0$ and $f(x_1, g(x_1)) = 0$, so

$$\begin{aligned} 0 &= f(x_1, g(x_1)) - f(x, g(x)) \\ &= f(x_1, g(x_1)) - f(x_1, g(x)) + f(x_1, g(x)) - f(x, g(x)) \\ &= \frac{\partial f}{\partial y}(x_1, \eta)(g(x_1) - g(x)) + \frac{\partial f}{\partial x}(\xi, g(x))(x_1 - x) \end{aligned}$$

for some ξ between x and x_1 and some η between $g(x)$ and $g(x_1)$. Therefore

$$\frac{g(x_1) - g(x)}{x_1 - x} = -\frac{\partial f}{\partial x}(\xi, g(x)) \Big/ \frac{\partial f}{\partial y}(x_1, \eta).$$

Hence

$$g'(x) = \lim_{x_1\to x} \frac{g(x_1) - g(x)}{x_1 - x} = -\frac{\partial f}{\partial x}(x, g(x)) \Big/ \frac{\partial f}{\partial y}(x, g(x)).$$

□

Appendix: Topological Theorems of Calculus

THEOREM 1. **[Intermediate Value Theorem]** *Let f be continuous on $[a, b]$. Suppose that $f(a) < 0$ and $f(b) > 0$, then there exists $a < c < b$ such that $f(c) = c$.*

THEOREM 2. **[Bounded Theorem]** *Let f be continuous on $[a, b]$. Then there exists constants α, β such that*

$$\forall\ x \in [a, b],\ \alpha \leq f(x) \leq \beta$$

THEOREM 3. **[Max/Min Theorem]** *Let f be continuous on $[a, b]$. Then there exists points $c_{max}, c_{min} \in [a, b]$ such that*

$$\forall\ x \in [a, b],\ f(c_{min}) \leq f(x) \leq f(c_{max})$$

THEOREM 4. **[Uniform Continuity]** *Let f be continuous on $[a, b]$. Then f is uniformly continuous on $[a, b]$. That is,*

$$\forall \epsilon > 0,\ \exists\ \delta > 0 \text{ such that } \forall x, y \in [a, b], |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon$$

THEOREM 5. **[Riemann Integrable]** *Let f be continuous on $[a, b]$. Then f is Riemann integrable on $[a, b]$. That is,*

$$\forall \epsilon > 0,\ \exists\ \mathcal{P} = \{a = t_0 < t_1 \cdots < t_n = b\} \text{ with } U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$$

THEOREM 6. **[Uniform Convergence]** *Let $\{f_n \mid n = 1, 2, \ldots\}$ be a sequence of functions which converge uniformly to f on $[a, b]$.*

- *If each f_n is continuous on $[a, b]$ then f is continuous on $[a, b]$*
- *If each f_n is integrable on $[a, b]$ then f is integrable on $[a, b]$, and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx \ = \ \int_a^n f(x)\, dx$$

- *If each f_n is differentiable on $[a, b]$ and f_n' converges uniformly to g on $[a, b]$, then f is differentiable, and $f'(x) = g(x)$. Otherwise said, the derivative of the limit equals the limit of the derivatives.*

Further reading

♣ "Introduction to Metric and Topological Spaces," by W. A. Sutherland is a nice introduction to topology. It is the standard book recently for Math 445, "Introduction to Topology". This makes a nice starting point for reading more about topology.

♢ "Topology - a first course" by James R. Munkres is more advanced, and covers much more territory than the Sutherland book. This book covers all the background material, and then some, usually assumed about topology in graduate school. This book contains the proof given in the sections on category of these notes.

♡ "Topology" by James Dugundji was the authoritative text for many years. It can be impenetrable at times, but pursues topics fearlessly - especially in the exercises. Read this book if you like reading Nietzche.

♠ "Topology" by John Hocking and Gail Young is the source book on topology for the 1960's generation - which included Dugundji. It includes many celebrated examples of pathological spaces, including the "Alexander's Horned Sphere" and lots of others, like space filling curves and such.