**Math 414 Analysis II**
Bonus Problem 2

Recall that $A \subseteq \mathbb{R}$ is nowhere dense if for all $a < b$ there are $a < c < d < b$ such that $A \cap (c, d) = \emptyset$.

We say that $B$ is *meager* if there are nowhere dense sets $A_1, A_2, \ldots$ such that $B = \bigcup_{n=1}^{\infty} A_n$. Last semester we prove the Baire Category Theorem that says that $\mathbb{R}$ is not meager (see §3.5 of Abbott).

Meagerness is another "notion of smallness".

a) Prove that if $A$ is meager and $B \subseteq A$, then $B$ is meager. Prove that if $A_1, A_2, \ldots$ are meager, then $\bigcup_{n=1}^{\infty} A_n$ is meager.

Every countable set is meager and we know that the Cantor set $C$ is nowhere dense, and hence meager.

We will show that "meager" and "measure zero" are different notions of smallness.

Suppose $\mathbb{Q} = \{a_1, a_2, \ldots\}$. Let

$$I_{i,j} = (a_i - \frac{1}{2^{i+j+1}}, a_i + \frac{1}{2^{i+j+1}})$$

for $i, j \in \mathbb{N}$. Let

$$G_j = \bigcup_{i=1}^{\infty} I_{i,j} \text{ and } B = \bigcap_{j=1}^{\infty} G_j.$$

b) Prove that $B$ has measure zero.

c) Prove $G_j^c$ is nowhere dense for each $j$.

d) Conclude that $B$ has measure zero but is not meager, while $B^c$ is meager but does not have measure zero.

Note we have shown that $\mathbb{R} = B \cup B^c$ is the union of a measure zero set and a meager set (two "small" sets–but in different senses of small.)