Most Continuous Functions are Nowhere Differentiable

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The Space of Continuous Functions

Let $K = [0,1]$ and let $\mathcal{C}(K)$ be the set of all continuous functions $f : K \to \mathbb{R}$.

Definition 1 For $f \in \mathcal{C}(K)$ we define $||f||$, the *norm* of f, by $||f|| = \sup\{|f(x)| : x \in K\}$.

Since K is compact and $|f|$ is continuous, $||f||$ is well-defined.

Exercise 2 Prove that $||f - g|| \leq ||f - h|| + ||h - g||$ for all $f, g, h \in \mathcal{C}(K)$

Definition 3 We say (f_n) is a *Cauchy sequence* in $\mathcal{C}(K)$ if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$||f_n - f_m|| < \epsilon$$

for all $n, m \geq N$.

Note that (f_n) converges to g uniformly if and only if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $||f_n - g|| < \epsilon$ for all $n \geq N$. Moreover, we know that any Cauchy sequence is uniformly convergent. Since the uniform limit of continuous functions is continuous, we have the following proposition.

Proposition 4 *If (f_n) is a Cauchy sequence in $\mathcal{C}(K)$, then there is $f \in \mathcal{C}(K)$ such that f_n converges to f.*

Definition 5 We say that $F \subseteq \mathcal{C}(K)$ is closed if every Cauchy sequence in F converges to an element of F.

Definition 6 If $f \in \mathcal{C}(K)$ and $\epsilon > 0$, let

$$B_\epsilon(f) = \{g \in \mathcal{C}(K) : ||f - g|| < \epsilon\}.$$

We call $B_\epsilon(f)$ the *open ball* of radius ϵ around f.

Similarly, we define

$$\overline{B}_\epsilon(f) = \{g \in \mathcal{C}(K) : ||f - g|| \leq \epsilon\}.$$

Lemma 7 *Each $\overline{B}_\epsilon(f)$ is closed.*

Proof Suppose (f_n) is a Cauchy sequence in $B_\epsilon(f)$. Suppose $f_n \to g$. If $x \in K$, then $|f_n(x) - f(x)| \leq \epsilon$ for all n. Hence $|g(x) - f(x)| \leq \epsilon$ for all x and $||g - f|| \leq \epsilon$.

Note that if $g \in B_\epsilon(f)$ and $0 < \delta \leq \epsilon - ||f - g||$, then $B_\delta(g) \subseteq B_\epsilon(f)$.

Definition 8 We say that $D \subseteq \mathcal{C}(K)$ is *dense* if $D \cap B_\epsilon(f)$ is nonempty for every open ball.

Intuitively, D is dense if every continuous function on K can be well-approximated by functions in D.

Definition 9 We say that $p : K \to \mathbb{R}$ is *piecewise-linear* if there is a partition $0 = a_0 < a_1 < \ldots < a_n = 1$ of $[0, 1]$ such that p is linear on the interval $[a_i, a_{i+1}]$ for $i = 0, \ldots, n$.

Let $PL(K) \subseteq \mathcal{C}(K)$ be the set of piecewise linear continuous functions on K.

Theorem 10 *$PL(K)$ is dense in $\mathcal{C}(K)$.*

Proof Suppose $f \in \mathcal{C}(K)$ and $\epsilon > 0$. Since K is compact, f is uniformly continuous on K. Thus there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/2$ for all $x, y \in K$ with $|x - y| < \delta$.

Let $0 = a_0 < a_1 < \ldots < a_n = 1$ be a partition of $[0, 1]$ such that $|a_{i+1} - a_i| < \delta$ for $i = 0, 1, \ldots$. Let $p : K \to \mathbb{R}$ be the piecewise linear function such that $p(a_i) = f(a_i)$ and p is linear on each $[a_i, a_{i+1}]$ for all i. If $x \in K$, there is an i such that $a_i \leq x \leq a_{i+1}$. Then

$$|p(x) - f(a_i)| = |p(x) - p(a_i)| \leq |p(a_{i+1}) - p(a_i)| < \epsilon/2$$

and $|f(x) - f(a_i)| < \epsilon/2$. Thus $|p(x) - f(x)| < \epsilon$ and $p \in B_\epsilon(f)$.

Baire Category in $\mathcal{C}(K)$

We begin by generalizing some concepts from $\mathbb{R}$ to $\mathcal{C}(K)$.

Definition 11 We say that $E \subseteq \mathcal{C}(K)$ is *nowhere dense* if for all open balls $B_\epsilon(f)$, there is an open ball $B_\delta(g) \subseteq B_\epsilon(f)$ with $D \cap B_\delta(g) = \emptyset$.

We say that $E \subseteq \mathcal{C}(K)$ is *meager* if

$$E = \bigcup_{n=1}^{\infty} E_n$$

where each E_n is nowhere dense. Meager sets are sometimes called sets of *first-category*.

Meager sets share some of the properties of measure zero sets.

Exercise 12 a) If E is meager and $F \subseteq E$, then F is meager.
b) If $E_1, E_2, \ldots$ are meager, then $E = \bigcup_{n=1}^{\infty} E_n$ is meager.

We think of meager sets as being "small". It is important to show that not every set is meager.

Theorem 13 (Baire Category Theorem for C(K)) *$\mathcal{C}(K)$ is not meager.*

Proof Suppose $E = \bigcup E_n$ where each $E_n \subseteq \mathcal{C}(K)$ is nowhere dense. We will find $f \in \mathcal{C}(K)$ with $f \notin E$ by constructing a sequence (f_n) converging uniformly to f as follows:

Let $f_0 \in \mathcal{C}(K)$. Let $\epsilon_0 = 1$. Given f_n and $\epsilon_n > 0$, since E_n is nowhere dense we can find $f_{n+1} \in B_{\epsilon_n}(f_n)$ and $\epsilon_{n+1} > 0$ such that:
i) $\overline{B}_{\epsilon_{n+1}}(f_{n+1}) \subseteq B_{\epsilon_n}(f_n)$;
ii) $B_{\epsilon_{n+1}}(f_{n+1}) \cap E_n = \emptyset$.

We claim that the sequence (f_n) is Cauchy. Let $\epsilon > 0$. Choose N such that $\epsilon_N < \epsilon$. If $n > m \geq N$, then $f_n \in \overline{B}_{\epsilon_m}(f_m)$. Hence $||f_n - f_m|| \leq \epsilon_m < \epsilon$. Thus there is $f \in \mathcal{C}(K)$ such that $f_n \to f$. Since $f_n \in \overline{B}_{\epsilon_m}(f_m)$ for all $n > m$, and, by Lemma 7, $\overline{B}_{\epsilon_m}(f_m)$ is closed, we know that $f \in \overline{B}_{\epsilon_m}(f_m)$ for all m. Since $\overline{B}_{\epsilon_m}(f_m) \cap E_m = \emptyset$, $f \notin E_m$ for any m. Thus $f \in \mathcal{C}(K) \setminus E$.

Exercise 14 Prove that every open or closed ball is nonmeager.

We give a useful characterization of nowhere dense closed sets.

Lemma 15 *Suppose $F \subseteq \mathcal{C}(K)$ is closed. The following are equivalent:*
i) F is nowhere dense;
ii) there is no open ball $B_\epsilon(f) \subseteq F$.

Proof i) $\Rightarrow$ ii) Clear.

ii) $\Rightarrow$ i) Suppose F is not nowhere dense. Then there is an open ball $B_\epsilon(f)$ such that every open ball $F \cap B_\delta(g) \neq \emptyset$ whenever $B_\delta(g) \subseteq B_\epsilon(f)$. We claim that $B_\epsilon(f) \subseteq F$. Let $g \in B_\epsilon(f)$. For each n we can find $f_n \in B_{1/n}(g) \cap E$. Then f_n converges uniformly to g. Hence $g \in E$. Thus $B_\epsilon(f) \subseteq E$.

Nowhere Differentiable Functions

Let $D = \{f \in \mathcal{C}(K) : f \text{ is differentiable at } x \text{ for some } x \in K\}$. We will prove that D is meager. By the Baire Category Theorem, this gives another proof that there are nowhere differentiable continuous functions. Indeed, it tells us that "most" continuous functions on K are nowhere differentiable!

Let $A_{n.m} =$

$$\left\{ f \in \mathcal{C}(K) : \text{ there is } x \in K \text{ such that } \left| \frac{f(t) - f(x)}{t - x} \right| \leq n \text{ if } 0 < |x - t| < \frac{1}{m} \right\}.$$

Lemma 16 *If $f \in D$, then $f \in A_{n,m}$ for some n and m.*

Proof Suppose f is differentiable at x. Choose n such that $|f'(x)| < n$. There is $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} \right| < n$$

if $0 < |t - x| < \delta$. Choose m such that $1/m < \delta$. Then $f \in A_{n,m}$.

Lemma 17 *Each $A_{n,m}$ is closed.*

Proof Suppose (f_i) is a Cauchy sequence in $A_{n,m}$ and $f_i \to f$. For each i we can find $x_i \in K$ such that

$$\left| \frac{f_i(t) - f_i(x_i)}{t - x} \right| \leq n \text{ for all } 0 < |x - t| < \frac{1}{m}.$$

By the Bolzono–Weierstrass Theorem (x_i) has a convergent subsequence. By replacing the sequence f_n by a subsequence, we may, without loss of generality, assume that (x_i) converges. Suppose x_i converges to x. Suppose $0 < |x - t| < \frac{1}{m}$. Then

$$\left|\frac{f(t) - f(x)}{t - x}\right| = \lim_{i \to \infty} \left|\frac{f_i(t) - f_i(x_i)}{t - x_i}\right| \leq n.$$

Hence $f \in A_{n,m}$.

Lemma 18 *Each $A_{n,m}$ is nowhere dense.*

Proof Since $A_{n,m}$ is closed, if suffices, by Lemma 15, to show that $A_{n,m}$ does not contain an open ball. Consider the open ball $B_\epsilon(f)$. We must find $g \in B_\epsilon(f)$ with $g \notin A_{n,m}$. By Theorem 10, we can find a piecewise linear $p(x)$ such that $||f - p|| < \epsilon/2$.

Since the graph of p is a finite union of line segments, p is differentiable at all but finitely many points and we can find $M \in \mathbb{N}$ such that $|p'(x)| \leq M$ for all x where p is differentiable. Choose $k > \frac{2(M+n)}{\epsilon}$.

There is a continuous piecewise linear function $\phi(x)$ where $|\phi(x)| \leq 1$ for all $x \in K$ and $\phi'(x) = \pm k$ for all x where k is differentiable. [Consider the partition $a_i = i/k$ for $i = 0, \ldots, k$ and let $\phi(a_i) = 0$ if i is even and 1 if i is odd.] Let

$$g(x) = p(x) + \frac{\epsilon}{2}\phi(x).$$

Since $||f - p|| < \epsilon/2$ and $||g - p|| < \epsilon/2$, $||f - g|| < \epsilon$.

We claim that $g \notin A_{n,m}$. Let $x \in [0, 1]$. If p and ϕ are differentiable at x, then

$$|g'(x)| = \left|p'(x) \pm \frac{\epsilon}{2}k\right|$$

Since $|p'(x)| \leq M$, $|g'(x)| > n$. In general, we can find $l > m$ such that $g|[x, x + \frac{1}{l}$ and $g|[x - \frac{1}{l}, x]$ are linear and the absolute value of the slope is greater than n. In particular, if $0 < |x - t| < \frac{1}{l} < \frac{1}{m}$, then

$$\left|\frac{g(t) - g(x)}{t - x}\right| > n$$

and $g \notin A_{n,m}$. Thus $B_\epsilon(f) \not\subseteq A_{n,m}$.

Theorem 19 *D is meager. In particular, there are continuous nowhere differentiable functions.*

Proof Since each $A_{n,m}$ is nowhere dense,

$$A = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m}$$

is meager. Since $D \subseteq A$, D is meager.

By the Baire Category Theorem for $\mathcal{C}(K)$, we know that $\mathcal{C}(K)$ is not meager. Thus there is $f \in \mathcal{C}(K)$ with $f \notin D$. Indeed, if we think of meager sets as being "small", this tells us that "most" $f \in \mathcal{C}(K)$ are nowhere differentiable.