Metric Spaces
Math 413 Honors Project

1 Metric Spaces

Definition 1.1 Let X be a set. A *metric* on X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

i) $d(x, y) = d(y, x)$;

ii) $d(x, y) = 0$ if and only if $x = y$;

iii) $d(x, y) \leq d(x, z) + d(z, y)$.

If d is a metric on X we call (X, d) a *metric space*.

We think of $d(x, y)$ as the *distance* from x to y.

Metric spaces arise in mathematics in many guises. Many of the basic properties of $\mathbb{R}$ that we will study in Math 413 are really properties of metric spaces and it is often useful to understand these ideas in full generality.

We already know some natural examples of metric spaces.

Exercise 1.2 [Euclidean metric] Suppose $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Let

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

.

a) For $n = 1$ show that $d(x, y) = |y - x|$ is a metric on $\mathbb{R}$. Note that d is also a metric on $\mathbb{Q}$.

In fact d is a metric on $\mathbb{R}^n$. The hard part is showing that iii) holds. For notational simplicity assume $n = 2$.

b) Show that $2xy \leq x^2 + y^2$ for any $x, y \in \mathbb{R}$.

c) (Schwartz Inequality) $|x_1y_1 + x_2y_2| \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$ [Hint: we may as well assume all the $x_i, y_i \geq 0$.]

d) Use the Schwartz inequlality to show that $\sqrt{(x_1+y_1)^2+(x_2+y_2)^2} \leq \sqrt{x_1^1+x_2^2}+\sqrt{y_1^2+y_2^2}$.

e) Show that d is a metric on $\mathbb{R}^2$. [Hint: Use d) and the fact that $x_i - y_i = (x_i - z_i) + (z_i - y_i)$.]

There are other interesting examples.

Exercise 1.3 [Discrete Spaces] Let X be any nonempty set. Define

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}.$$

Prove that d is a metric.

Exercise 1.4 [Taxi Cab Metric] For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ define

$$d(\mathbf{x}, \mathbf{y}) = \sum |x_i - y_i|.$$

Prove that d is a metric.

Why is this called the *taxi cab metric* in $\mathbb{R}^2$?

Exercise 1.5 [Sequence Spaces] Let A be any nonempty set and let Seq_A be the set of all infinite sequences $(a_1, a_2, \ldots)$ where each $a_i \in A$. If $\mathbf{a} = (a_1, a_2, \ldots)$ and $\mathbf{b} = (b_1, b_2, \ldots)$ are in Seq_A define

$$d(\mathbf{a}, \mathbf{b}) = \begin{cases} 0 & \text{if } a_n = b_n \text{ for all } n \\ \frac{1}{n} & \text{if } n \text{ is least such that } a_n \neq b_n \end{cases}.$$

Prove that d is a metric on Seq_A.

Metric spaces also arise naturally in number theory.

Exercise 1.6 [p-adic metric] Let p be a prime number. If m is a nonzero natural number let $v_p(m)$ be the largest number j such that p^j divides m.

a) Suppose $m_1, m_2, n_1, n_2 \in \mathbb{N}$, $n_1, n_2 \neq 0$ and $\frac{m_1}{n_1} = \frac{m_2}{n_2}$. Show that $m_1 - n_1 = m_2 - n_2$.

This allows us to extend v_p to $\mathbb{Q}$ by defining $v_p(\frac{m}{n}) = v_p(m) - v_p(n)$.

Define $d_p : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by

$$d_p(x,y) = \begin{cases} 0 & \text{if } x = y \\ p^{-v_p(x-y)} & \text{otherwise} \end{cases}.$$

b) Prove that d_p is a metric on $\mathbb{Q}$.

Exercise 1.7 A metric is said to be *non-Archimedean* if

$$d(x, y) \leq \max(d(x, z), d(y, z))$$

for all $x, y, z \in X$, otherwise it is called Archimedean. For each of the metrics above decide it is Archimedean or non-Archimedean.

Sequences in Metric Spaces

Definition 1.8 Suppose (X, d) is a metric space. Suppose $x_i \in X$ for $i \in \mathbb{N}$. Then we call $(x_1, x_2, \ldots)$ a *sequence* in X.

We say that $(x_i)_{i=1}^{\infty}$ converges to $x \in X$ if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x, x_n) < \epsilon$ for all $n \geq N$.

Exercise 1.9 a) Suppose $(x_n)_{i=1}^{\infty}$ is a sequence in $\mathbb{R}^n$. Prove that (x_n) converges to x in the Euclidean metric if and only if x_n converges to x in the taxi cab metric.

b) Let X be any set and let d be the discrete metric on X given in Exercise 1.3. Give a simple characterization of the convergent sequences.

c) Suppose A is nonempty and d is the metric on Seq_A given in Exercise 1.5. For $i \in \mathbb{N}$ let $\mathbf{a}_i = (a_{i,1}, a_{i,2}, a_{i,3}, \ldots) \in \mathrm{Seq}_A$. Prove that the sequence $(\mathbf{a}_1, \mathbf{a}_2, \ldots)$ converges if and only if for each j there is N_j such that $a_{i,j} = a_{N_j,j}$ for all $i \geq N_j$. [Note: Think about this carefully $(\mathbf{a}_1, \mathbf{a}_2, \ldots)$ is a sequence where each $\mathbf{a}_i$ is also a sequence!]

Exercise 1.10 † Recall the definition of the metric spaces $(\mathbb{Q}, d_p)$ from Exercise 1.6

a) Prove that the sequence $(p, p^2, p^3, \ldots)$ converges to 0 in the metric space $(\mathbb{Q}, d_p)$.

b) Find a sequence $(a_1, a_2, \ldots)$ such that no a_i is 0, yet (a_n) converges to zero in $(\mathbb{Q}, d_p)$ for every prime p.

c) Prove that the sequence

$$(1 + p, 1 + p + p^2, 1 + p + p^2 + p^3, 1 + p + p^2 + p^3 + p^4, \ldots)$$

converges to $\frac{1}{1-p}$.

d)) Let $p = 5$. Consider the sequence $(4, 34, 334, 3334, 33334, \ldots)$ and prove that it converges to $\frac{2}{3}$ in the metric space $(\mathbb{Q}, d_5)$.

Definition 1.11 Suppose (X, d) is a metric space. We say that a sequence $(x_n)_{n=1}^{\infty}$ is *Cauchy* if for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Exercise 1.12 Let (X, d) be a metric space. Prove that every convergent sequence is Cauchy.

Definition 1.13 We say that a metric space (X, d) is *complete* if every Cauchy sequence converges.

Exercise 1.14 a) Prove that every discrete metric space (as in Exercise 1.3) is complete.

b) Prove that $\mathbb{R}^n$ with the taxi-cab metric is complete.

c) Prove that $\mathbb{R}^n$ with the Euclidean metric is complete. [Hint: One way to do this is to use b) and Exercise 1.9.] Note that $\mathbb{Q}$ with the Euclidean metric is not complete.

d) Prove that for any nonempty set A the space (Seq_A, d) of Exercise 1.5 is complete.

Exercise 1.15 † Let p be prime. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence in the metric space $(\mathbb{Q}, d_p)$. Prove that $\lim a_n = 0$ if and only if the sequence

$$(a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots)$$

is Cauchy.

Exercise 1.16 †† Let $p = 5$. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence of integers such that

i) $a_n \equiv a_{n+1} (\bmod\ 5^n)$ and

ii) $a_n^2 + 1 \equiv 0 (\bmod\ 5^n)$

for all $n \in \mathbb{N}$ (where $a \equiv b (\bmod\ M)$ means that M divides $b - a$).

a) Prove that such a sequence is Cauchy in the metric space $(\mathbb{Q}, d_5)$.

b) Suppose (a_n) converges to l in $(\mathbb{Q}, d_5)$ prove that $l^2 = -1$. Conclude that (a_n) is divergent in $(\mathbb{Q}, d_5)$

It remains to show that there is such a sequence. Let $a_1 = 2$.

c) Given that $a_1, \ldots, a_n$ such that i) and ii) hold show that there is $b \in \mathbb{Z}$ such that i) and ii) hold for $a_{n+1} = a_n + b5^n$. Conclude that $(\mathbb{Q}, d_p)$ is not complete.

2 Topology of Metric Spaces

Let X be a metric space.

Definition 2.1 For $r > 0$ and $x \in X$ we define the *open ball* of radius r around x,

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

We say that $O \subseteq X$ is *open* if and only for every $x \in O$ there is $\epsilon > 0$ such that $B_\epsilon(x) \subseteq O$.

Exercise 2.2 a) Prove that every open ball is open.

b) Prove that $\{x : d(x, a) > r\}$ is an open set for each $a \in X$ and $r > 0$.

Definition 2.3 Let $A \subseteq X$. We say that x is a *limit point* of A if for all $\epsilon > 0$ there is $y \in B_\epsilon(x)$ with $x \neq y$. Let $\lim(A)$ be the limit points of A.

We say that $F \subseteq X$ is *closed* if and only if every limit point of F is in F, i.e., $\lim(F) \subseteq F$.

Exercise 2.4 b) Suppose O_i is open for all $i \in I$. Prove that $\displaystyle\bigcup_{i\in I} O_i$ is open.

c) Suppose $O_1, \ldots, O_n$ are open. Prove that $O_1 \cap \ldots \cap O_n$ is open.

d) Prove that F is closed if and only if F^c is open. Conclude that the arbitrary intersection of closed sets is closed and that a finite union of closed sets is closed.

e) Prove that F is closed if and only if for any sequence $(x_n)_{n=1}^\infty$ if $\lim x_n = x$ and each $x_n \in F$, then $\lim x_n \in F$.

Definition 2.5 Let (X, d_X) and (Y, d_Y) are metric spaces. Then $f : X \to Y$ is continuous if and only if for all $a \in X$ for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x \in X$ if $d_X(a, x) < \delta$, then $d_Y(f(x) - f(a)) < \epsilon$.

Exercise 2.6 Suppose X, Y are a metric spaces and $f : X \to Y$. Prove that the following are equivalent:

a) f is continuous;

b) if $O \subseteq Y$ is open, then $f^{-1}(O)$ is open;

c) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed.

Definition 2.7 If $A \subseteq X$, an *open cover* of A is a collection of open set $(O_i : i \in I)$ such that $A \subseteq \bigcup_{i\in I} O_i$. A *subcover* is a finite set $I_0 \subseteq I$ such that $A \subseteq \bigcup_{i\in I_0} O_i$.

Definition 2.8 $K \subseteq X$ is *compact* if and only if every open cover of K has a finite subcover.

Exercise 2.9 Suppose $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ are nonempty compact subsets of X. Prove that $\bigcap_{n=1}^{\infty} K_n$ is nonempty. [Hint: Consider the open sets $O_i = K_i^c$].

Definition 2.10 We say that $A \subseteq X$ is *bounded* if there is an open ball $B_r(x) \supseteq A$.

Exercise 2.11 Prove that if A and B are bounded, then $A \cup B$ is bounded.

Exercise 2.12 Suppose $K \subseteq X$ is compact.

a) Prove that K is bounded.

b) Prove that K is closed. [Hint: If $a \notin K$, let $O_n = \{x : d(x, a) > \frac{1}{n}\}$. Start by showing $O_1, O_2, \ldots$ is an open cover.]

c) Suppose X is an infinite discrete metric space. Prove that X is closed and bounded but not compact.

Exercise 2.13 Suppose X, Y are metric spaces, $f : X \to Y$ is continuous and $K \subseteq X$ is compact. Prove that $f(K)$ is compact.

Exercise 2.14 Let X be a compact metric space. Prove that X is complete.

Definition 2.15 If X is a metric space, we say that $A \subseteq X$ is *dense* if and only if for any $x \in X$ and any $\epsilon > 0$ there is $a \in A \cap B_\epsilon(x)$.

We say that X is *separable* if there is a countable dense set.

Exercise 2.16 Give an example of a nonseparable metric space.

Exercise 2.17 Suppose X is a compact metric space.

a) Prove that for every $\epsilon > 0$ there is $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ such that $X = B_\epsilon(x_1) \cup \ldots \cup B_\epsilon(x_n)$ (i.e., X can be covered by finitely many balls of radius ϵ.)

b) Prove that every compact metric space is separable.

Exercise 2.18 † Suppose X is a compact metric space and Y is a metric space. Let $C(X)$ be the set of all continuous $f : X \to \mathbb{R}$. If $f, g \in C(X)$ define

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

a) Prove that $(C(X), d)$ is a metric space.

b) Prove that $(C(X), d)$ is a complete metric space.

Exercise 2.19 [Contraction Mapping Theorem] Suppose X is a nonempty complete metric space, $f : X \to X$ and there is a real number $0 < c < 1$ such that

$$d(f(x), f(y)) \leq c d(x, y)$$

for all $x, y \in X$. Prove that f is continuous and there is a unique $\alpha \in X$ such that $f(\alpha) = \alpha$. [Hint: Follow the steps of Exercise 4.3.9 from the text.]

3 Constructing $\mathbb{R}$

What is $\mathbb{R}$? So far, rather than answering this question, we have taken an axiomatic approach. We began 413 by giving the axioms for a complete ordered fields. The only facts we have used about $\mathbb{R}$ are consequences of the axioms for complete ordered fields. Our goal in these exercises is to prove the following theorem.

Theorem 3.1 *i)* [existence] *There is a complete ordered field.*
ii) [uniqueness] *Any two complete ordered fields are isomorphic.*

This shows that our axiomatic approach has not cost us anything. Since the reals are the only complete ordered field, anything we want to prove about the reals is also true for all complete ordered fields.

Our approach to Theorem 3.1 will be in the spirit of Kroeneker's statement "*God made the natural numbers, all else is the work of man.*" In Math 215 you saw how to start with $\mathbb{N}$ and construct define $\mathbb{Z}$ and $\mathbb{Q}$. Now we will show how, starting with the ordered field of rational numbers $\mathbb{Q}$, we can construct the reals.

Definition 3.2 We say that $C \subseteq \mathbb{Q}$ is a *Dedekind cut* if:
i) for all $x \in C$ there is $y \in C$ such that $x < y$;
ii) if $x \in C$ and $y < x$, then $y \in C$;
iii) there is $y \notin C$ (i.e., $C \neq \mathbb{Q}$).
Let $\mathcal{D} = \{C : C \text{ is a Dedekind cut}\}$.

Exercise 3.3 a) If $q \in \mathbb{Q}$, the $\{x : x < q\}$ is a Dedekind cut.
b) Show that $\{x : x \leq 0 \text{ or } x^2 < 2\}$ is a Dedekind cut.
c) Suppose K is a complete ordered field. If $r \in K$, let $C_r = \{q \in \mathbb{Q} : q < r\}$. Prove that $r \mapsto C_r$ is a bijection between K and $\mathcal{D}$.

Exercise 3.4 Suppose $A, B \in \mathcal{D}$. Let

$$A + B = \{a + b : a \in A, b \in B\}.$$

a) Prove that $A + B$ is a Dedekind cut.
b) Prove that $(\mathcal{D}, +)$ is an Abelian group. What is the zero element? What is $-C$ for $C \in \mathcal{D}$?

Exercise 3.5 Suppose $A, B \in \mathcal{D}$. Define $A < B$ if there is $b \in B$ such that $b > a$ for all $a \in A$.

a) Prove that $(\mathcal{D}, <)$ is a linear order.[1]

b) Suppose $A < B$. Prove that $A + C < B + C$.

c) Suppose $X \subseteq \mathcal{D}$ is nonempty and bounded above. Prove that

$$\bigcup_{A \in X} A$$

is a least upper bound for X. Thus $(\mathcal{D}, <)$ is a complete linear order.

Exercise 3.6 a) Suppose $A, B \in \mathcal{D}$, $A > 0$ and $B > 0$. Let

$$A \cdot B = \{x : \exists a \in A \exists b \in B \ a > 0, b > 0 \text{ and } x \leq ab\}.$$

a) Prove

$$A \cdot B = \{x : x \leq 0 \text{ or } \exists a \in A \exists b \in B \ a > 0, b > 0 \text{ and } x = ab\}.$$

b) Prove that $A \cdot B$ is a Dedekind cut.

c) Give a reasonable definition of $A \cdot B$ for arbitrary A, B. Prove that with this definition $(\mathcal{D}, +, \cdot)$ is a field.

d) Show that if $A < B$ and $C > 0$ then $A \cdot C < B \cdot C$.

e) Conclude that $(\mathcal{D}, +, \cdot, <)$ is a complete ordered field.

We have now proved that there is a complete ordered field.

Exercise 3.7 Suppose $(K, +, \cdot, <)$ is an ordered field with additive identity 0 and multiplicative identity 1.

a) Let $\phi_0 : \mathbb{N} \to K$ be the function

$$\phi_0(x) = \underbrace{1 + \ldots + 1}_{x\text{−times}}$$

and let $\phi_1(x) : \mathbb{Z} \to K$ be the function

$$\phi_1(x) = \begin{cases} \phi_0(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\phi_0(x) & \text{if } x < 0 \end{cases}.$$

[1] Prove that $A \not< A$, if $A < B$ and $B < C$, then $A < C$, and for all A, B either $A = B$ of $A < B$ or $B < A$.

Prove that ϕ_1 is an order preserving ring homomorphism.

b) Let $\phi : \mathbb{Q} \to K$ be the function

$$\phi\left(\frac{m}{n}\right) = \phi_1(m)/\phi_1(n).$$

Prove that ϕ is an order preserving (and hence one-to-one) ring homomorphism.

Thus we can view $\mathbb{Q}$ as an ordered subfield of any ordered field K

Exercise 3.8 Suppose F and K are ordered fields. Let $\psi : F \to K$ be defined by $\psi(a) = b$ if and only if

$$\{q \in \mathbb{Q} : x < a \text{ in } F\} = \{q \in \mathbb{Q} : x < b \text{ in } K\}.$$

a) Prove that ψ is a well-defined function.

b) Prove that ψ is an order preserving isomorphism between F and K.

We have now prove Theorem 3.1.

There is another way to construct a complete ordered field.

Let $\mathcal{C} = \{(a_n) : (a_n) \text{ is a Cauchy sequence of rational numbers}\}$. Define $(a_n) \sim (b_n)$ if and only if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|a_n - b_m| < \epsilon$ for all $n, m > N$.

For $x \in \mathcal{C}$ let $x/\sim = \{y \in \mathcal{C} : x \sim y\}$ and let

$$\mathcal{X} = \{x/\sim : x \in \mathcal{C}\}.$$

Exercise 3.9 Define $+, \cdot$ and $<$ to make $(\mathcal{X}, +, \cdot, <)$ a complete ordered field. Conclude that $(\mathcal{X}, +, \cdot, <)$ and $(\mathcal{D}, +, \cdot, <)$ are isomorphic ordered fields.