Lebesgue’s Characterization of Riemann Integrable Functions

Measure Zero Sets

Read

- Abbott §7.6 pg 203–207
- Chapter 2 §1 & 2 of Quick Tour of the Topology of \( \mathbb{R} \) on measure zero sets.

Sets of Discontinuity

In Math 413 we proved that for any \( f : \mathbb{R} \to \mathbb{R} \) the set of points where \( f \) is discontinuous is an \( F_\sigma \)-set. Let’s review the key steps of that proof.

**Definition 7** For \( f : [a, b] \to \mathbb{R} \) let

\[
D = \{ x \in [a, b] : f \text{ is discontinuous at } x \}.
\]

If \( \epsilon > 0 \), let

\[
D_\epsilon = \{ x : \text{ for all } \delta > 0 \text{ there are } y, z \in (x-\delta, x+\delta) \text{ with } |f(y) - f(z)| \geq \epsilon \}.
\]

**Lemma 8** \( D = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}} \).

**Proof** Suppose \( x \in D \). Then there is \( \epsilon > 0 \) such that for any \( \delta > 0 \) there is a \( y \) such that \( |x - y| < \delta \) and \( |f(x) - f(y)| \geq \epsilon \). If \( \frac{1}{n} < \epsilon \), then \( x \in D_{\frac{1}{n}} \).

Thus \( D \subseteq \bigcup_{n=1}^{\infty} D_{\frac{1}{n}} \).

On the other hand, suppose \( x \in D_{\frac{1}{m}} \). For all \( n \) choose

\[
y_n, z_n \in (x - \frac{1}{n}, x + \frac{1}{n})
\]

such that \( |f(y_n) - f(z_n)| \geq \frac{1}{m} \). Note that \( \lim y_n = \lim z_n = x \). If \( f \) were continuous at \( x \), then \( \lim f(y_n) = \lim f(z_n) = x \) and there would be an \( N \) such that \( |f(y_n) - f(x_n)| < \frac{1}{m} \) for all \( n > N \), a contradiction. Thus each \( D_{\frac{1}{n}} \subseteq D \) and

\[
D = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}.
\]
Lemma 9 Each $D_\epsilon$ is closed.

Proof Suppose $x \not\in D_\epsilon$. Then there is $\delta > 0$ such that $|f(y) - f(z)| < \epsilon$ for all $y, z \in (x - \delta, x + \delta)$. It is easy to see that if $y \in (x - \delta, x + \delta)$ then $y \not\in D_\epsilon$. Thus $D_\epsilon^c$ is open and $D_\epsilon$ is closed.

Corollary 10 $D$ is an $F_\sigma$-set.

Theorem 11 (Lebesgue’s Theorem) Suppose $f : [a, b] \to \mathbb{R}$ is bounded. Then $f$ is Riemann-integrable if and only if $D$ has measure zero.

Proof

$(\Rightarrow)$ Suppose $f$ is integrable. To prove that $D$ has measure zero it suffices to prove that each $D_{1/m}$ has measure zero.

Let $P = \{x_0, \ldots, x_n\}$ be a partition such that $U(f, P) - L(f, P) < \frac{\epsilon}{2m}$. Let $O_i$ be the interval $(x_i - \frac{\epsilon}{4(n+1)}, x_i + \frac{\epsilon}{4(n+1)})$. Then $|O_i| = \frac{\epsilon}{2(n+1)}$.

Let $B = \{i : (x_{i-1}, x_i) \cap D_{1/m} \neq \emptyset\}$. If $i \in B$ there are $x, y \in (x_{i-1}, x_i)$ with $|f(x) - f(y)| \geq \frac{1}{m}$. Thus $M_i - m_i \geq \frac{1}{m}$. Thus

$$\sum_{i \in B} \frac{1}{m} \Delta x_i \leq U(f, P) - L(f, P) < \frac{\epsilon}{2m}$$

and $\sum_{i \in B} \Delta x_i < \frac{\epsilon}{2}$. If $x \in D_{1/m}$, then either $x \in (x_{i-1}, x_i)$ for some $i \in B$, or $x = x_i$ for some $i = 0, \ldots, n$. Thus

$$D_{1/m} \subseteq O_0 \cup \ldots \cup O_n \cup \bigcup_{i \in B} (x_{i-1}, x_i)$$

and

$$|O_0| + \cdots + |O_n| + \sum_{i \in B} \Delta x_i < (n + 1) \frac{\epsilon}{2(n + 1)} + \frac{\epsilon}{2} = \epsilon.$$ 

Thus $D_{1/m}$ has measure zero and $D$ has measure zero.

$(\Leftarrow)$ Suppose $D$ has measure zero. Suppose $|f(x)| < M$ for all $x \in [a, b]$.

Let $\epsilon > 0$. Choose $n$ such that $\frac{\epsilon}{2} < \frac{\epsilon}{M}$. The set $D_{1/n} \subseteq D$ has measure zero. Thus we can find a countable collection of open intervals $O_1, O_2, \ldots$ such that

$$D_{1/n} \subseteq \bigcup_{j=1}^{\infty} O_j \text{ and } \sum_{j=1}^{\infty} |O_j| < \frac{\epsilon}{4M}.$$
Since \( D_n \) is a closed subset of \([a, b]\) it is compact. Thus, by the Heine–Borel Theorem, we can find \( j_1, \ldots, j_m \) such that
\[
D_n \subseteq O_{j_1} \cup \cdots \cup O_{j_m}.
\]
Let \( O = O_{j_1} \cup \cdots \cup O_{j_m} \). If \( x \in [a, b] \setminus O \), then \( x \not\in D_n \). Thus there is an open interval \( I_x \) with \( x \in I_x \) such that \(|f(y) - f(z)| < \frac{1}{n} \) for all \( y, z \in I_x \). Note that
\[
[a, b] \subseteq O_{j_1} \cup \cdots \cup \bigcup_{x \in O} I_x.
\]
By one more application of compactness, we can find \( x_1, \ldots, x_k \) such that
\[
[a, b] \subseteq O_{j_1} \cup \cdots \cup O_{j_m} \cup I_{x_1} \cup \cdots \cup I_{x_k}.
\]
We choose a partition \( P \) that contains all of the endpoints of \( O_{j_1}, \ldots, O_{j_m}, I_{x_1}, \ldots, I_{x_k} \). Each interval \([x_{i-1}, x_i]\) is either contained in \( O \) or \(|f(z) - f(y)| < \frac{1}{n} \) for all \( y, z \in [x_{i-1}, x_i] \). In the later case \( M_i - m_i \leq \frac{1}{n} \). Let \( B = \{ i : [x_{i-1}, x_i] \subseteq O \} \). Then
\[
U(f, P) - L(f, P) = \sum_{i \in B} (M_i - m_i) \Delta x_i + \sum_{i \notin B} (M_i - m_i) \Delta x_i \\
\leq \sum_{i \in B} 2M \Delta x_i + \sum_{i \notin B} \frac{1}{n} \Delta x_i \\
\leq 2M(|O_{j_1} + \cdots + O_{j_m}|) + \frac{1}{n}(b - a) \\
< 2M\frac{\epsilon}{4M} + \frac{\epsilon}{2} < \epsilon.
\]
Thus \( f \) is integrable.