Lebesgue's Characterization of Riemmann Integrable Functions

Measure Zero Sets

Read

- Abbott §7.6 pg 203–207
- Chapter 2 §1 & 2 of *Quick Tour of the Topology of* $\mathbb{R}$ on measure zero sets.

Sets of Discontinuity

In Math 413 we proved that for any $f : \mathbb{R} \to \mathbb{R}$ the set of points where f is discontinuous is an F_σ-set. Let's review the key steps of that proof.

Definition 7 For $f : [a, b] \to \mathbb{R}$ let

$$D = \{x \in [a, b] : f \text{ is discontinuous at } x\}.$$

If $\epsilon > 0$, let

$$D_\epsilon = \{x : \text{ for all } \delta > 0 \text{ there are } y, z \in (x - \delta, x + \delta) \text{ with } |f(y) - f(z)| \geq \epsilon\}.$$

Lemma 8 $D = \displaystyle\bigcup_{n=1}^{\infty} D_{\frac{1}{n}}$.

Proof Suppose $x \in D$. Then there is $\epsilon > 0$ such that for any $\delta > 0$ there is a y such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$. If $\frac{1}{n} < \epsilon$, then $x \in D_{\frac{1}{n}}$. Thus $D \subseteq \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}$.

On the other hand, suppose $x \in D_{\frac{1}{m}}$. For all n choose

$$y_n, z_n \in (x - \frac{1}{n}, x + \frac{1}{n})$$

such that $|f(y_n) - f(z_n)| \geq \frac{1}{m}$. Note that $\lim y_n = \lim z_n = x$. If f were continuous at x, then $\lim f(y_n) = \lim f(z_n) = x$ and there would be an N such that $|f(y_n) - f(x_n)| < \frac{1}{m}$ for all $n > N$, a contradiction. Thus each $D_{\frac{1}{m}} \subseteq D$ and

$$D = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}.$$

Lemma 9 *Each D_ϵ is closed.*

Proof Suppose $x \notin D_\epsilon$. Then there is $\delta > 0$ such that $|f(y) - f(z)| < \epsilon$ for all $y, z \in (x - \delta, x + \delta)$. It is easy to see that if $y \in (x - \delta, x + \delta)$ then $y \notin D_\epsilon$. Thus D_ϵ^c is open and D_ϵ is closed.

Corollary 10 *D is an F_σ-set.*

Theorem 11 (Lebesgue's Theorem) *Suppose $f : [a, b] \to \mathbb{R}$ is bounded. Then f is Riemann-integrable if and only if D has measure zero.*

Proof

($\Rightarrow$) Suppose f is integrable. To prove that D has measure zero it suffices to prove that each $D_{\frac{1}{m}}$ has measure zero.

Let $P = \{x_0, \ldots, x_n\}$ be a partition such that $U(f, P) - L(f, P) < \frac{\epsilon}{2m}$. Let O_i be the interval $(x_i - \frac{\epsilon}{4(n+2)}, x_i + \frac{\epsilon}{4(n+2)})$. Then $|O_i| = \frac{\epsilon}{2(n+1)}$.

Let $B = \{i : (x_{i-1}, x_i) \cap D_{\frac{1}{m}} \neq \emptyset\}$. If $i \in B$ there are $x, y \in (x_{i-1}, x_i)$ with $|f(x) - f(y)| \geq \frac{1}{m}$. Thus $M_i - m_i \geq \frac{1}{m}$. Thus

$$\sum_{i \in B} \frac{1}{m} \Delta x_i \leq U(f, P) - L(f, P) < \frac{\epsilon}{2m}$$

and $\sum_{i \in B} \Delta x_i < \frac{\epsilon}{2}$. If $x \in D_{\frac{1}{m}}$, then either $x \in (x_{i-1}, x_i)$ for some $i \in B$, or $x = x_i$ for some $i = 0, \ldots, n$. Thus

$$D_{\frac{1}{m}} \subseteq O_0 \cup \ldots \cup O_n \cup \bigcup_{i \in B} (x_{i-1}, x_i)$$

and

$$|O_0| + \cdots + |O_n| + \sum_{i \in B} \Delta x_i < (n+1) \frac{\epsilon}{2(n+1)} + \frac{\epsilon}{2} = \epsilon.$$

Thus $D_{\frac{1}{m}}$ has measure zero and D has measure zero.

($\Leftarrow$) Suppose D has measure zero. Suppose $|f(x)| < M$ for all $x \in [a, b]$.

Let $\epsilon > 0$. Choose n such that $\frac{b-a}{n} < \frac{\epsilon}{2}$. The set $D_{\frac{1}{n}} \subseteq D$ has measure zero. Thus we can find a countable collection of open intervals $O_1, O_2, \ldots$ such that

$$D_{\frac{1}{n}} \subseteq \bigcup_{j=1}^{\infty} O_j \text{ and } \sum_{j=1}^{\infty} |O_j| < \frac{\epsilon}{4M}.$$

Since $D_{\frac{1}{n}}$ is a closed subset of $[a, b]$ it is compact. Thus, by the Heine–Borel Theorem, we can find $j_1, \ldots, j_m$ such that

$$D_{\frac{1}{n}} \subseteq O_{j_1} \cup \ldots \cup O_{j_m}.$$

Let $O = O_{j_1} \cup \ldots \cup O_{j_m}$. If $x \in [a, b] \setminus O$, then $x \notin D_{\frac{1}{n}}$. Thus there is an open interval I_x with $x \in I_x$ such that $|f(y) - f(z)| < \frac{1}{n}$ for all $y, z \in I_x$. Note that

$$[a, b] \subseteq O_{j_1} \cup \ldots \cup \bigcup_{x \notin O} I_x.$$

By one more application of compactness, we can find $x_1, \ldots, x_k$ such that

$$[a, b] \subseteq O_{j_1} \cup \ldots \cup O_{j_m} \cup I_{x_1} \cup \ldots \cup I_{x_k}.$$

We choose a partition P that contains all of the endpoints of $O_{j_1}, \ldots, O_{j_m}, I_{x_1}, \ldots, I_{x_k}$. Each interval $[x_{i-1}, x_i]$ is either contained in O or $|f(z) - f(y)| < \frac{1}{n}$ for all $y, z \in [[x_{i-1}, x_i]$. In the later case $M_i - m_i \leq \frac{1}{n}$. Let $B = \{i : [x_{i-1}, x_i] \subseteq O\}$. Then

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i \in B} (M_i - m_i)\Delta x_i + \sum_{i \notin B} (M_i - m_i)\Delta x_i \\
&\leq \sum_{i \in B} 2M\Delta x_i + \sum_{i \notin B} \frac{1}{n}\Delta x_i \\
&\leq 2M(|O_{j_1} + \ldots + O_{j_m}|) + \frac{1}{n}(b - a) \\
&< 2M\frac{\epsilon}{4M} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Thus f is integrable.