Math 414 Analysis II
Problem Set 11

Due Friday April 16

1) Prove that the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly to e^x on every closed interval $[-M, M]$.

2) a) Calculate $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$.

[Hint: Consider the power series $f(x) = \sum x^n$ and $f'(x)$.]

b) Calculate $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{2^n n!}$.

3) Suppose $\alpha \in \mathbb{R}$ We define $\binom{\alpha}{0} = 1$ and

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

for $n = 1, 2, \ldots$.

For example

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}}{3!} = \frac{1}{16}.$$

Let $f(x) = (1+x)^\alpha$.

a) Prove that

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

is the Taylor series for f centered at 0.

b) Prove that this series coverges if $|x| < 1$.

c) Prove that the series converges uniformly to $(1+x)^\alpha$ for $x \in [0, M]$ for $0 \le M < 1$.[Notice that if $t \ge 0$ then $(1+t)^{\alpha-n-1} \le 1$ for large enough n.][1]

d) Find the Taylor series for

$$g(x) = \frac{1}{\sqrt{1-x^2}} \text{ and } h(x) = \arcsin(x)$$

and argue that these series converge to the desired functions on (-1,1).[Hint: Use the fact that we know the Taylor series for $(1+x)^{1/2}$ on (-1,1).]

4) By considering Taylor series decide which of the following function is largest, and which is smallest, for $x > 0$ near 0.

$$f(x) = 1 + \sin x, \quad g(x) = e^x, \quad or \quad h(x) = \frac{1}{\sqrt{1-2x}}.$$

[1] **Bonus Problem**: Prove that the series converges uniformly to $(1+x)^\alpha$ on $[-c, c]$ for $0 < c < 1$.]