Math 414 Analysis II
Problem Set 11

Due Friday April 16

1) Prove that the power series \(\sum_{n=0}^{\infty} \frac{x^n}{n!}\) converges uniformly to \(e^x\) on every closed interval \([-M, M]\).

2) a) Calculate \(\sum_{n=1}^{\infty} \frac{n}{2^n}\).
   [Hint: Consider the power series \(f(x) = \sum x^n\) and \(f'(x)\).]

   b) Calculate \(\sum_{n=0}^{\infty} \frac{2n+1}{2^n n!}\).

3) Suppose \(\alpha \in \mathbb{R}\) We define \(\binom{\alpha}{n}\) = 1 and
   \[
   \binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}
   \]
   for \(n = 1, 2, \ldots\).
   For example
   \[
   \binom{\frac{1}{3}}{3} = \frac{1}{3} \cdot \frac{-1}{2} \cdot \frac{-3}{4} = \frac{1}{16}.
   \]
   Let \(f(x) = (1 + x)^{\alpha}\).
   a) Prove that
   \[
   \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n
   \]
   is the Taylor series for \(f\) centered at 0.
   b) Prove that this series converges if \(|x| < 1\).
   c) Prove that the series converges uniformly to \((1 + x)^{\alpha}\) for \(x \in [0, M]\) for \(0 \leq M < 1\).
   Notice that if \(t \geq 0\) then \((1 + t)^{\alpha - n} - 1 \leq 1\) for large enough \(n\).\(^1\)
   d) Find the Taylor series for
   \[
   g(x) = \frac{1}{\sqrt{1 - x^2}} \quad \text{and} \quad h(x) = \arcsin(x)
   \]
   and argue that these series converge to the desired functions on \((-1,1)\).\(^{2}\)[Hint: Use the fact that we know the Taylor series for \((1 + x)^{1/2}\) on \((-1,1)\).]

4) By considering Taylor series decide which of the following function is largest, and which is smallest, for \(x > 0\) near 0.
   \[
   f(x) = 1 + \sin x, \quad g(x) = e^x, \quad \text{or} \quad h(x) = \frac{1}{\sqrt{1 - 2x}}
   \]

\(^1\)Bonus Problem: Prove that the series converges uniformly to \((1 + x)^{\alpha}\) on \([-c, c]\) for \(0 < c < 1\).