Math 414 Analysis II
Problem Set 2

Due Friday January 30

Do the following problems from Abbott's *Understanding Analysis*
Excercise 7.4.1
Exercise 7.4.4

1) We construct an example of an integrable function with infinitely many discontinuities.

a) Let $f_k : [0,1] \to \mathbb{R}$ be the function

$$f_k(x) = \begin{cases} 1 & \text{if } x \leq \frac{1}{k} \text{ or } x = \frac{1}{n} \text{ for } n = 1, 2, \ldots k-1 \\ 0 & \text{otherwise} \end{cases}.$$

Prove that f_k is integrable and $\int_0^1 f_k = \frac{1}{k}$.

Let $f : [0,1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for } n = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}.$$

Note that $f(x) \leq f_k(x)$ for all k. We will prove that f is integrable.

b) Prove that $L(f, P) = 0$ for all partitions.

c) Suppose $\epsilon > 0$. Find a partition P such that $U(f, P) < \epsilon$ [Hint: First find a k such that $\int_0^1 f_k < \frac{\epsilon}{2}$.]

d) Conclude that f is integrable and $\int_0^1 f = 0$.