Math 414 Analysis II
Problem Set 4

Due Friday February 13

1) Let t be the Thomae function. Find a continuous function $h : [0,1] \to \mathbb{R}$ such that $h(x) \geq t(x)$ for all $x \in [0,1]$ and $\int_0^1 h < \frac{1}{2}$. [Of course we know that for any $\epsilon > 0$ we can find such an h with $\int_0^1 h < \epsilon$.]

2) Let $F : (0, +\infty) \to \mathbb{R}$ be the function

$$F(x) = \int_0^x \frac{1}{1+t^2}\, dt + \int_0^{\frac{1}{x}} \frac{1}{1+t^2}\, dt.$$

Prove that F is constant.

3) (Improper Integrals) If $f : [a, +\infty) \to \mathbb{R}$ we let

$$\int_a^\infty f(t)\, dt = \lim_{x \to \infty} \int_a^x f(t)\, dt$$

if the limit exists.

a) Calculate $\int_1^\infty \frac{1}{t^r}\, dt$ for $r > 1$.

b) Let $H(x) = \int_1^x \frac{1}{t}\, dt$. Prove that $H(2^n) = nH(2)$. Conclude that $\int_1^\infty \frac{1}{t}\, dt$ does not exist. [Hint: Consider Exercise 7.5.4].

c) Suppose $f(x) \geq 0$ for all $x \geq a$ and $\int_a^\infty f(t)\, dt$ exists. Suppose $0 \leq g(x) \leq f(x)$ for all $x \geq a$ and g is integrable on $[0, r]$ for all r. Prove that $\int_a^\infty g(t)\, dt$ exists. [Hint: Show $G(x) = \int_0^x g(t)\, dt$ is nondecreasing and bounded above.]

d) Use c) to show that $\int_0^\infty \frac{1}{t^2+1}\, dt$ exists.

e) Use the fact that $(\arctan(x))' = \frac{1}{x^2+1}$ to calculate $\int_0^\infty \frac{1}{t^2+1}\, dt$.