Supplementary Lecture Notes on Integration
Math 414

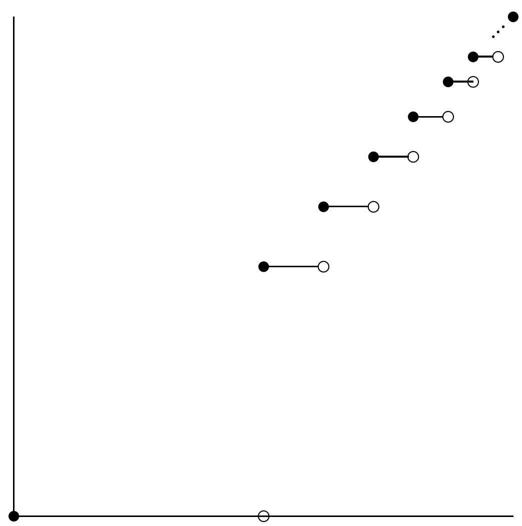
Spring 2004

Integrable Functions with Many Discontinuities

We give several examples of integrable functions with many discontinuities.

We showed for homework that every nondecreasing function $f : [a, b] \to \mathbb{R}$ is integrable. It is not hard to construct a nondecreasing function with countably many discontinuities. For example, let $f : [0, 1] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 1 - \frac{1}{n} & \text{if } 1 - \frac{1}{n} \leq x < 1 - \frac{1}{n+1} \end{cases} .$$



Then f is nondecreasing and hence integrable. Clearly f is discontinuous at $1 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

Next consider the Thomae function $t : [0, 1] \to \mathbb{R}$

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \text{ where } m \neq 0 \text{ and } n \text{ are relatively prime} \end{cases}.$$

We have shown that t is continuous at every irrational number but discontinuous at every rational number.

Proposition 1 *The Thomae function is integrable and* $\int_0^1 t = 0$.

Proof The irrational numbers are dense. Thus for any partition $P = \{x_0 \ldots, x_n\}$ there is always an irrational in every interval $[x_{i-1}, x_i]$. Thus $L(t, P) = 0$. To prove that t is integrable it is enough to show that for every $\epsilon > 0$ there is a partition P_ϵ with $U(t, P) < \epsilon$.

Let $A_n = \{x : t(x) \geq \frac{1}{n}\}$. If $x \in A_n$, then $x = i/j$ where $i, j \leq n$. In particular A_n is finite.

Suppose $\epsilon > 0$. Choose n such that $\frac{1}{n} < \frac{\epsilon}{2}$. We will choose a partition P_ϵ such that each point of A_n is in an interval $[x_{i-1}, x_i]$ where

$$\Delta x_i = x_i - x_{i-1} < \frac{\epsilon}{2|A_n|}.$$

Let $B = \{i : A_n \cap [x_{i-1}, x_i] = \emptyset\}$. Note that $|B| \leq |A_n|$. If $i \in B_i$ then $M_i < \frac{1}{n} < \frac{\epsilon}{2}$, while if $i \notin B_i$, then $M_i = 1$. Thus

$$\begin{aligned} U(t, P_\epsilon) &= \sum_{i \in B} M_i \Delta x_i + \sum_{i \notin B} M_i \Delta x_i \\ &< \sum_{i \in B} \frac{\epsilon}{2} \Delta x_i + \sum_{i \notin B} \Delta x_i \\ &< \frac{\epsilon}{2} + |A_n| \frac{\epsilon}{2|A_n|} \\ &< \epsilon. \end{aligned}$$

Since $U(t, P_\epsilon) - L(t, P_\epsilon) < \epsilon$, t is integrable. Since

$$0 = L(t, P_\epsilon) \leq \int_0^1 t \leq U(t, P_\epsilon) < \epsilon$$

for all ϵ, $\int_0^1 t = 0$.

The Thomae function has countably many discontinuities. We next give an example of an integrable function with uncountably many discontinuities.

Let C be the Cantor set and let $f : [0,1] \to \mathbb{R}$. Be the function

$$f(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}.$$

Proposition 2 *f is integrable and $\int_0^1 f = 0$.*

Recall the construction of the Cantor set from *Quick Tour of the Topology of* $\mathbb{R}$ §11 or section 3.1 of Abbott. We build a sequence of closed sets $C_0 \supset C_1 \supset C_2 \supset \ldots$ as follows:

$$C_0 = [0,1],$$

we throw out the middle third and are left with

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1],$$

we throw out the middle third of each interval and get

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \ldots$$

In general C_n is a union of 2^n closed intervals of length $\frac{1}{3^n}$.

The Cantor set $C = \bigcup_{n=0}^{\infty} C_n$ is an uncountable, closed, nowhere dense set.

Let $f_n : [0,1] \to \mathbb{R}$ be the function

$$f_n(x) = \begin{cases} 1 & \text{if } x \in C_n \\ 0 & \text{if } x \notin C_n \end{cases}.$$

Note that $f(x) \leq f_n(x)$ for all $x \in [0,1]$. The function f_n is easy to integrate. $f_n(x) = 1$ on 2^n intervals of length $\frac{1}{3^n}$ and is 0 everywhere else. Thus

$$\int_0^1 f_n = \left(\frac{2}{3}\right)^n.$$

Suppose $\epsilon > 0$. Choose n such that $(2/3)^n < \frac{\epsilon}{2}$ and choose a partition P such that $U(f_n, P) - L(f_n, P) < \frac{\epsilon}{2}$. Then $U(f_n, P) < \epsilon$. But $f(x) \leq f_n(x)$ for all $x \in [0,1]$. Thus

$$U(f, P) \leq U(f_n, P) < \epsilon.$$

Clearly $L(f,P) \geq 0$ (indeed since C is nowhere dense $L(f,P) = 0$). Thus $U(f,P) - L(f,P) < \epsilon$. Hence f is integrable.

Since for any $\epsilon > 0$ there is a partition P with $U(f,P) < \epsilon$ we must have $\int_0^1 f = 0$.

Exercise 3 Show that f is continuous at x if and only if $x \notin C$. Thus f has uncountably many discontinuities.

Approximating Integrable Functions

Lemma 4 *Suppose $f : [a,b] \to \mathbb{R}$ is bounded, P is a partition of $[a,b]$ and $\epsilon > 0$. There is a continuous function $h : [a,b] \to \mathbb{R}$ such that $f(x) \leq h(x)$ for all $x \in [a,b]$ and*

$$\int_a^b h - U(f,P) < \epsilon.$$

Proof We begin by finding a step function $\widehat{h}$ such that $f(x) \leq \widehat{h}(x)$ for all $x \in [a,b]$ and $\int_a^b \widehat{h}(x) = U(f,P)$. Suppose $P = \{x_0, \ldots, x_n\}$.

Let $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and let

$$\widehat{h}(x) = \begin{cases} M_i & \text{if } x_{i-1} < x < x_i \\ M_n & \text{if } x = b \end{cases}.$$

Then $f(x) \leq \widehat{h}(x)$ for all $x \in [a,b]$ and

$$\int_a^b \widehat{h} = \sum_{i=1}^n M_i \Delta x_i = U(f,P).$$

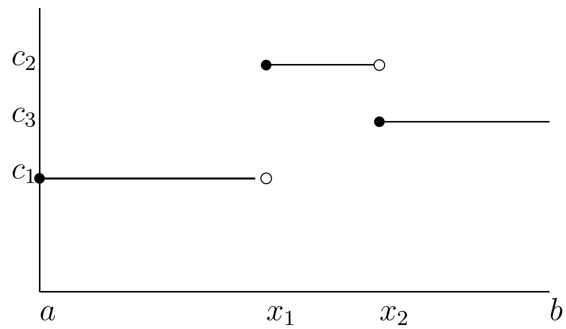
We next find a continuous function $h \geq \widehat{h}$ with $\int_a^b (h - \widehat{h}) < \epsilon$. We do this by modifying $\widehat{h}$ near the points $x_1, \ldots, x_{n-1}$ where there may be a discontinuity.

The idea of the proof is easy but the notation can get messy. Rather than giving a detailed proof we give an example.

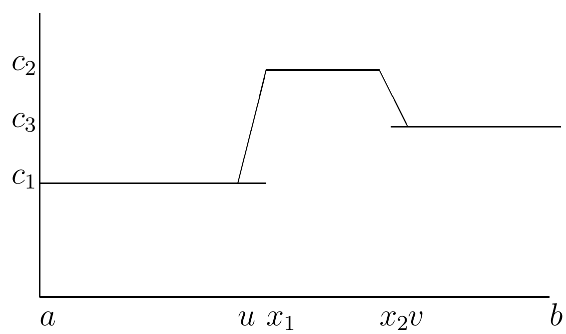
Suppose $P = \{a, x_1, x_2, b\}$ and

$$\widehat{h}(x) = \begin{cases} c_1 & a \leq x < x_1 \\ c_2 & x_1 \leq x < x_2 \\ c_3 & x_2 \leq x \leq b \end{cases}$$

where $c_1 < c_2$ and $c_2 > c_3$.



We want a continuous function $h \geq \widehat{h}$ that looks like this:



We will choose u and v so that $\int_a^b (h - \widehat{h}) < \epsilon$. We do this by making sure both of the triangles have area less than $\frac{\epsilon}{2}$.

Suppose $x_1 - u < \frac{\epsilon}{c_2 - c_1}$ and $v - x_2 < \frac{\epsilon}{c_2 - c_3}$. Then each triangle has area less than $\frac{\epsilon}{2}$ and $\int_a^b (h - \widehat{h}) < \epsilon$.

In this case

$$h(x) = \begin{cases} c_1 & a \leq x \leq u \\ c_1 + \frac{c_2 - c_1}{x_1 - u}(x - u) & u < x < x_1 \\ c_2 & x_1 \leq x \leq x_2 \\ c_2 + \frac{c_3 - c_2}{v - x_2}(v - x) & x_2 < x \leq v \\ c_3 & v < x \leq b \end{cases}.$$

It is clear that $h(x) \geq \widehat{h}(x)$ for all $x \in [a, b]$ and $\int_a^b (h - \widehat{h}) < \epsilon$. Thus $h(x) \geq f(x)$ for all $x \in [a, b]$ and $\int_a^b h - U(f, P) < \epsilon$.

Similarly we can find a continuous $g \leq f$ such that $L(f,P) - \int_a^b g < \epsilon$.

Corollary 5 *If $f : [a,b] \to \mathbb{R}$ is Riemann–integrable, then for any $\epsilon > 0$ there are continuous functions $g, h : [a,b] \to \mathbb{R}$ such that $g(x) \leq f(x) \leq h(x)$ for all $x \in [a,b]$ and $\int_a^b (h-g) < \epsilon$.*

Proof Since f is integrable we can find a partition P such that

$$U(f,P) - L(f,P) < \frac{\epsilon}{3}.$$

By the Lemma we can find continuous functions g and h such that $g(x) \leq f(x) \leq h(x)$ for all $x \in [a,b]$,

$$\int_a^b h - U(f,P) < \frac{\epsilon}{3} \text{ and } L(f,P) - \int_a^b g < \frac{\epsilon}{3}.$$

Then $\int_a^b h - g < \epsilon$.

Another Characteriztation of Integrable Functions

We prove Theorem 8.1.2 from Abbott's *Understanding Analysis*

Theorem 6 *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann-integrable with*

$$\int_a^b f = A$$

if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|R(f,P) - A| < \epsilon$$

for any δ-fine tagged partition P.

Proof
($\Rightarrow$) We first prove this in case f is continuous. We begin as in the proof that continuous functions are integrable.

Let $\epsilon > 0$. By Uniform Continuity there is $\delta > 0$ such that if $x, y \in [a,b]$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Choose a partition $P = \{x_0, \ldots, x_n\}$ and tags $z_1, \ldots, z_n$ such that

$$\Delta x_i = x_i - x_{i-1} < \delta$$

for $i = 1, \ldots, n$. Since f is continuous there are $u, v \in [x_{i-1}, x_i]$ with $f(u) = m_i$ and $f(v) = M_i$. By choice of δ, $|M_i - m_i| < \frac{\epsilon}{b-a}$. Thus

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \frac{\epsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \epsilon.$$

But

$$L(f,P) \leq R(f,P) \leq U(f,P)$$

and

$$L(f,P) \leq \int_a^b f \leq U(f,P).$$

Thus

$$\left| R(f,P) - \int_a^b f \right| < \epsilon.$$

We now consider the general case where f may not be continuous. Let $\epsilon > 0$. We know that there are continuous functions $g, h : [a,b] \to \mathbb{R}$ such that $g(x) \leq f(x) \leq h(x)$ for all $x \in [a,b]$ and

$$\left| \int_a^b h - \int_a^b g \right| < \frac{\epsilon}{2}.$$

By the argument above we can find $\delta > 0$ such that for any tagged δ-fine partition P

$$\left| R(h,P) - \int_a^b h \right| < \frac{\epsilon}{2} \text{ and } \left| R(g,P) - \int_a^b g \right| < \frac{\epsilon}{2}.$$

Since

$$R(g,P) \leq R(f,P) \leq R(h,P)$$

and

$$\int_a^b g \leq \int_a^b f \leq \int_a^b h,$$

$$R(g,P) - \int_a^b h \leq R(f,P) - \int_a^b f \leq R(h,P) - \int_a^b g$$

and

$$\left|R(f,P)-\int_a^b f\right| \leq \max\left(\left|R(g,P)-\int_a^b h\right|, \left|R(h,P)-\int_a^b g\right|\right).$$

But

$$\left|R(g,P)-\int_a^b h\right| \leq \left|R(g,P)-\int_a^b g\right| + \left|\int_a^b h - \int_a^b g\right| \leq \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

and

$$\left|R(h,P)-\int_a^b g\right| \leq \left|R(h,P)-\int_a^b h\right| + \left|\int_a^b h - \int_a^b g\right| \leq \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Thus

$$\left|R(g,P)-\int_a^b h\right| \leq \epsilon$$

as desired.

($\Leftarrow$) We first need one claim.

Claim For any bounded function $f:[a,b]\to\mathbb{R}$, any partition P and any $\epsilon>0$ we can find a tagging such that

$$U(f,P)-R(f,P)<\epsilon.$$

Note: if f is continous we can do better, we could choose $z_i\in[x_{i-1},x_i]$ such that $z_i=M_i$ and then $R(f,P)=U(f,P)$. If f is not continuous, it is possible that there is no $z_i\in[x_{i-1},x_i]$ with $f(z_i)=M_i$. Even if this is not possible, we can find $z_i\in[x_{i-1},x_i]$ with $f(z_i)$ as close as we'd like to M_i.

For $i=1,\ldots,n$ choose $z_i\in[x_{i-1},x_i]$ with

$$M_i-f(z_i)<\frac{\epsilon}{n(x_i-x_{i-1})}.$$

Then

$$\begin{aligned} U(f,P)-R(f,P) &= \sum_{i=1}^n (M_i-f(z_i))\Delta x_i \\ &< \sum_{i=1}^n \frac{\epsilon}{n(x_i-x_{i-1})}\Delta x_i \\ &= \sum_{i=1}^n \frac{\epsilon}{n}=\epsilon. \end{aligned}$$

as desired. This proves the claim.

We can now finish the proof. We want to show that f is integrable and $\int_a^b f = A$.

Let $\epsilon > 0$. There is $\delta > 0$ such that

$$|R(f, P) - A| < \frac{\epsilon}{4}$$

for any tagged δ-fine partition P.

By the claim we can find a tagging P_1 of P such that

$$U(f, P) - R(f, P_1) < \frac{\epsilon}{4}.$$

Similarly, we can find a tagging P_2 of P such that

$$R(f, P_2) - L(f, P) < \frac{\epsilon}{4}.$$

Note that

$$|R(f, P_1) - R(f, P_2)| \leq |R(f, P_1) - A| + |R(f, R_2) - A| < \frac{\epsilon}{2}$$

and

$$U(f, P) - L(f, P) \leq U(f, P) - R(f, P_1) + R(f, P_2) - L(f, P) + |R(f, P_1) - R(f, P_2)| < \epsilon.$$

Thus f is integrable.

These arguments also show that for any $\epsilon > 0$ we can find a partiion P with

$$|U(f, P) - A| < \epsilon.$$

Thus $\int_a^b f = A$.

Lebesgue's Characterization of Riemmann Integrable Functions

Measure Zero Sets

Read

- Abbott §7.6 pg 203–207
- Chapter 2 §1 & 2 of *Quick Tour of the Topology of* $\mathbb{R}$ on measure zero sets.

Sets of Discontinuity

In Math 413 we proved that for any $f : \mathbb{R} \to \mathbb{R}$ the set of points where f is discontinuous is an F_σ-set. Let's review the key steps of that proof.

Definition 7 For $f : [a, b] \to \mathbb{R}$ let

$$D = \{x \in [a, b] : f \text{ is discontinuous at } x\}.$$

If $\epsilon > 0$, let

$$D_\epsilon = \{x : \text{ for all } \delta > 0 \text{ there are } y, z \in (x-\delta, x+\delta) \text{ with } |f(y)-f(z)| \geq \epsilon\}.$$

Lemma 8 $D = \displaystyle\bigcup_{n=1}^{\infty} D_{\frac{1}{n}}$.

Proof Suppose $x \in D$. Then there is $\epsilon > 0$ such that for any $\delta > 0$ there is a y such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$. If $\frac{1}{n} < \epsilon$, then $x \in D_{\frac{1}{n}}$. Thus $D \subseteq \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}$.

On the other hand, suppose $x \in D_{\frac{1}{m}}$. For all n choose

$$y_n, z_n \in (x - \frac{1}{n}, x + \frac{1}{n})$$

such that $|f(y_n) - f(z_n)| \geq \frac{1}{m}$. Note that $\lim y_n = \lim z_n = x$. If f were continuous at x, then $\lim f(y_n) = \lim f(z_n) = x$ and there would be an N such that $|f(y_n) - f(x_n)| < \frac{1}{m}$ for all $n > N$, a contradiction. Thus each $D_{\frac{1}{m}} \subseteq D$ and

$$D = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}.$$

Lemma 9 *Each D_ϵ is closed.*

Proof Suppose $x \notin D_\epsilon$. Then there is $\delta > 0$ such that $|f(y) - f(z)| < \epsilon$ for all $y, z \in (x - \delta, x + \delta)$. It is easy to see that if $y \in (x - \delta, x + \delta)$ then $y \notin D_\epsilon$. Thus D_ϵ^c is open and D_ϵ is closed.

Corollary 10 *D is an F_σ-set.*

Theorem 11 (Lebesgue's Theorem) *Suppose $f : [a, b] \to \mathbb{R}$ is bounded. Then f is Riemann-integrable if and only if D has measure zero.*

Proof

($\Rightarrow$) Suppose f is integrable. To prove that D has measure zero it suffices to prove that each $D_{\frac{1}{m}}$ has measure zero.

Let $P = \{x_0, \ldots, x_n\}$ be a partition such that $U(f, P) - L(f, P) < \frac{\epsilon}{2m}$. Let O_i be the interval $(x_i - \frac{\epsilon}{4(n+2)}, x_i + \frac{\epsilon}{4(n+2)})$. Then $|O_i| = \frac{\epsilon}{2(n+1)}$.

Let $B = \{i : (x_{i-1}, x_i) \cap D_{\frac{1}{m}} \neq \emptyset\}$. If $i \in B$ there are $x, y \in (x_{i-1}, x_i)$ with $|f(x) - f(y)| \geq \frac{1}{m}$. Thus $M_i - m_i \geq \frac{1}{m}$. Thus

$$\sum_{i \in B} \frac{1}{m} \Delta x_i \leq U(f, P) - L(f, P) < \frac{\epsilon}{2m}$$

and $\sum_{i \in B} \Delta x_i < \frac{\epsilon}{2}$. If $x \in D_{\frac{1}{m}}$, then either $x \in (x_{i-1}, x_i)$ for some $i \in B$, or $x = x_i$ for some $i = 0, \ldots, n$. Thus

$$D_{\frac{1}{m}} \subseteq O_0 \cup \ldots \cup O_n \cup \bigcup_{i \in B} (x_{i-1}, x_i)$$

and

$$|O_0| + \cdots + |O_n| + \sum_{i \in B} \Delta x_i < (n+1) \frac{\epsilon}{2(n+1)} + \frac{\epsilon}{2} = \epsilon.$$

Thus $D_{\frac{1}{m}}$ has measure zero and D has measure zero.

($\Leftarrow$) Suppose D has measure zero. Suppose $|f(x)| < M$ for all $x \in [a, b]$.

Let $\epsilon > 0$. Choose n such that $\frac{b-a}{n} < \frac{\epsilon}{2}$. The set $D_{\frac{1}{n}} \subseteq D$ has measure zero. Thus we can find a countable collection of open intervals $O_1, O_2, \ldots$ such that

$$D_{\frac{1}{n}} \subseteq \bigcup_{j=1}^{\infty} O_j \text{ and } \sum_{j=1}^{\infty} |O_j| < \frac{\epsilon}{4M}.$$

Since $D_{\frac{1}{n}}$ is a closed subset of $[a, b]$ it is compact. Thus, by the Heine–Borel Theorem, we can find $j_1, \ldots, j_m$ such that

$$D_{\frac{1}{n}} \subseteq O_{j_1} \cup \ldots \cup O_{j_m}.$$

Let $O = O_{j_1} \cup \ldots \cup O_{j_m}$. If $x \in [a, b] \setminus O$, then $x \notin D_{\frac{1}{n}}$. Thus there is an open interval I_x with $x \in I_x$ such that $|f(y) - f(z)| < \frac{1}{n}$ for all $y, z \in I_x$. Note that

$$[a, b] \subseteq O_{j_1} \cup \ldots \cup \bigcup_{x \notin O} I_x.$$

By one more application of compactness, we can find $x_1, \ldots, x_k$ such that

$$[a, b] \subseteq O_{j_1} \cup \ldots \cup O_{j_m} \cup I_{x_1} \cup \ldots \cup I_{x_k}.$$

We choose a partition P that contains all of the endpoints of $O_{j_1}, \ldots, O_{j_m}, I_{x_1}, \ldots, I_{x_k}$. Each interval $[x_{i-1}, x_i]$ is either contained in O or $|f(z) - f(y)| < \frac{1}{n}$ for all $y, z \in [[x_{i-1}, x_i]$. In the later case $M_i - m_i \leq \frac{1}{n}$. Let $B = \{i : [x_{i-1}, x_i] \subseteq O\}$. Then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i \in B} (M_i - m_i)\Delta x_i + \sum_{i \notin B} (M_i - m_i)\Delta x_i \\ &\leq \sum_{i \in B} 2M\Delta x_i + \sum_{i \notin B} \frac{1}{n}\Delta x_i \\ &\leq 2M(|O_{j_1} + \ldots + O_{j_m}|) + \frac{1}{n}(b - a) \\ &< 2M\frac{\epsilon}{4M} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Thus f is integrable.