Math 414–Midterm 2 Study Guide

• The midterm exam will be on Friday April 23.

• The exam will chapter 6 and the material on the contraction mapping theorem and existence and uniqueness theorems for differential equations. The course web page contains a week-by-week syllabus that gives a more detailed description of the material you are responsible for.

`http://www.math.uic.edu/~marker/math414/wtow.html`

• You should be comfortable with all of the key definitions. You should be able to state, apply and sketch the proof of all of the key results. The course web page contains a list of key concepts and key results.

`http://www.math.uic.edu/~marker/math414/concepts.html`

Sample Midterm

1) Let $f_1, f_2, \ldots$ and f be functions with domain A. Define the following concepts.

a) (f_n) converges to f pointwise on A;

b) (f_n) converges to f uniformly on A;

c) $\displaystyle\sum_{n=1}^{\infty} f_n$ converges to f pointwise on A.

2) State and prove the Weierstrass M-test.

3) Decide if the following statements are true or false. If false, give a counterexample or proof that the statement is false.

a) Suppose K is compact and $f_n : A \to \mathbb{R}$ for $n = 1, 2, \ldots$. If (f_n) converges pointwise to f, then f_n converges uniformly to f.

b) If $f_n : [0,1] \to \mathbb{R}$ for $n = 1, 2, \ldots$, (f_n) converges uniformly to f on $[0,1]$ and each f_n is integrable, then f is integrable.

c) If $f_n : [0,1] \to \mathbb{R}$ for $n = 1, 2, \ldots$, (f_n) converges uniformly to f on $[0,1]$ and each f_n is differentiable, then f is differentiable.

d) If $\sum_{n=0}^{\infty} a_n x^n$ converges at $M > 0$, then the series converges for each $x \in [-M, M]$.

e) The Taylor series for $\sin x$ converges uniformly to $\sin x$ on $\mathbb{R}$.

4) a) Suppose f is uniformly continuous on $\mathbb{R}$. Define a sequence of functions by $f_n(x) = f(x + \frac{1}{n})$. Show that $f_n \to f$ uniformly.

b) Show this fails if we only assume that f is continuous.

5) We say that a function f is *even* if $f(-x) = f(x)$ for all x. Suppose $f : \mathbb{R} \to \mathbb{R}$ is even and f is given by the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Prove that $a_n = 0$ for all odd n.

6) Let $f_n(x) = \dfrac{x}{1 + nx^2}$. Prove that f_n converges uniformly on $\mathbb{R}$.