The Weierstass Function

Spring 2004

In thesse notes we will fill in the details of the proof of the existence of a continuous $g : \mathbb{R} \to \mathbb{R}$ that is nowhere differentiable.

Let $h : \mathbb{R} \to \mathbb{R}$ be the function $h(x) = |y|$ where $-1 < y \leq 1$ and $x = 2n + y$ for some $n \in \mathbb{Z}$. In other words, h is a periodic function with period 2 that agrees with absolute value on $[-1, 1]$. For $n = 0, 1, 2, \ldots$ let

$$h_n(x) = \frac{1}{2^n} h(2^n x).$$

Then h_n is a function with period 2^{n-1}. The function h_n is nondifferentiable at points of the form $p/2^n$ where $p \in \mathbb{Z}$, but is differentiable at all other points. Indeed h_n is linear on each inteval $[p/2^n, p + 1/2^n]$.

For $m = 0, 1, \ldots$ let

$$g_m(x) = h_0(x) + h_1(x) + \ldots + h_m(x) \text{ and } g(x) = \sum_{i=0}^{\infty} h_i(x)$$

Lemma 1 *i) The series $\sum_{i=0}^{\infty} h_i(x)$ converges for all $x \in \mathbb{R}$. Thus g is a well defined functions.*

ii) The sequence of functions (g_m) converges uniformly to g.

Proof i) Since $|h(x)| \leq 1$, $|h_i(x)| \leq \frac{1}{2^i}$ for all x. Thus $\sum h_i(x)$ coverges for all $x \in \mathbb{R}$.

ii) For any x,

$$|g(x) - g_n(x)| \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}.$$

Given $\epsilon > 0$, choose N such that $\epsilon > \frac{1}{2^N}$. If $n \geq N$, then $|g(x) - g_n(x)| < \epsilon$ for all $x \in \mathbb{R}$. Thus $(g_n) \to g$ uniformly.

Corollary 2 *The function g is continuous.*

Proof Since each h_i is continuous, each g_m is continuous. Since $(g_m) \to g$ uniformly, g is continuous.

The proof that g is nowhere differentiable breaks into two cases.

Definition 3 We say that $x \in \mathbb{R}$ is a *dyadic rational* if $x = p/2^n$ for some $p \in \mathbb{Z}$ and $n = 0, 1, 2, \ldots$.

Lemma 4 *If $x \in \mathbb{R}$ is a dyadic rational, then g is not differentiable at x.*

Proof Let $x = \frac{p}{2^l}$ and let $x_n = x + \frac{1}{2^n}$ for $n \in \mathbb{N}$. Suppose $n > l$. Then

$$\begin{aligned} g(x_n) &= \sum_{i=0}^{\infty} \frac{1}{2^i} h\left(2^i\left(\frac{p}{2^l} + \frac{1}{2^n}\right)\right) \\ &= \sum_{i=0}^{l} \frac{1}{2^i} h\left(\frac{p}{2^{l-i}} + \frac{1}{2^{n-i}}\right) + \sum_{i=l+1}^{n} \frac{1}{2^i} h(2^{i-n}) \\ &= \sum_{i=0}^{l} \frac{1}{2^i} \left(h(p/2^{l-i}) + \delta_i\right) + \frac{n-l}{2^n} \end{aligned}$$

where $\delta_i = h(p/2^{l-i} + \frac{1}{2^{n-i}}) - h(p/2^{l-i})$. Looking at the definition of h, it is easy to see that $|h(x+\epsilon) - h(x)| \leq \epsilon$ for all $x \in \mathbb{R}$. Thus

$$\begin{aligned} g(x_n) &= \sum_{i=0}^{l} \frac{1}{2^i} h(p/2^{l-i}) + \sum_{i=1}^{n} \frac{1}{2^i} \delta_i + \frac{n-l}{2^n} \\ &= g(x) + \frac{n-l}{2^n} + \delta \end{aligned}$$

where

$$|\delta| \leq \sum_{i=1}^{n} \frac{1}{2^i} |\delta_i| \leq \frac{l+1}{2^n}.$$

Thus

$$n - 2l - 1 \leq \frac{g(x_n) - x}{x_n - x} \leq n + 1.$$

Thus

$$\lim_{n \to \infty} \frac{g(x_n) - x}{x_n - x} = \infty$$

and g is not differentiable at x.

Lemma 5 *Suppose x is not a dyadic rational. Each g_m is differentiable at x and $|g'_{m+1}(x) - g'_m(x)| = 1$.*

Proof Each h_i is differentiable at x. Thus each g_m is differentiable at x. Note that

$$|g'_{m+1}(x) - g'_m(x)| = |h'_{m+1}(x)|$$

but $h'_i(x) = \frac{1}{2^i}h'(2^i(x))(2^i) = h'(2^i x)$. But $h'(x) = \pm 1$ for all $x \notin \mathbb{Z}$. Thus $|g'_{m+1}(x) - g'_m(x)|$.

Lemma 6 *If $x \in \mathbb{R}$ is not a dyadic rational, then g is not differentiable at x.*

Proof For each m we can find a $p \in \mathbb{Z}$ such that

$$\frac{p}{2^m} < x < \frac{p+1}{2^m}.$$

Let $x_m = p/2^m$ and $y_m = \frac{p+1}{2^m}$. Then (x_m) and (y_m) both converge to x.

Consider the function g_m. We know this function is linear on the interval $[x_m, y_m]$. We also know that $g(x_m) = g_m(x_m)$ and $g(y_m) = g_m(y_m)$. On the other hand, since x is not a dyadic rational, we know that $g(x) > g_m(x)$. It follows that

$$\frac{g(y_m) - g(x)}{y_m - x} < \frac{g_m(y_m) - g_m(x)}{y_m - y} = g'_m(x) = \frac{g_m(x) - g_m(x_m)}{x - x_m} < \frac{g(x) - g(x_m)}{x - x_m}.$$

Suppose g is differentiable at x. Then for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that if $m \geq N$, then $g'(x)$ is within $\epsilon/2$ of $\frac{g(x)-g(x_m)}{x-x_m}$ and $\frac{g(y_m)-g(x)}{y_m-x}$ for all $m \geq N$. But from the inequalities above it would follow that $g'_N(x)$ and $g'_{N+1}(x)$ are within $\epsilon/2$ of $g'(x)$ and hence within ϵ of each other. This contradicts Lemma5. Thus g is not differentiable at x.

We have now proved the following result.

Theorem 7 (Weierstrass) *There is a continuous function $g : \mathbb{R} \to \mathbb{R}$ that is nowhere differentiable.*