Math 435 Number Theory I
Problem Set 3

Due: Friday September 16:

1) Prove that there are infinitely many prime numbers of the form $6n+5$.

2) a) Suppose x, y are integers. Prove that $x^2 - y^2$ is either odd or divisible by 4. [Hint: factor].

b) Suppose N is either odd or divisible by 4. Prove that

$$X^2 - Y^2 = N$$

has an integral solution.

c) (bonus problem) Prove further that $X^2 - Y^2 = N$ has a unique solution in the nonnegative integers if and only if $|N|$ or $|N|/4$ is either 1 or an odd prime.

3) Prove that $1 + 1/2 + \ldots + 1/n$ is not an integer if $n > 1$.
[Hint: Note that

$$1 + 1/2 + \ldots 1/n = \frac{\sum_{i=1}^{n} \frac{\operatorname{lcm}(1,\ldots,n)}{i}}{\operatorname{lcm}(1,\ldots,n)}.$$

Show

$$\sum_{i=1}^{n} \frac{\operatorname{lcm}(1,\ldots,n)}{i}$$

is odd. Consider the highest power of 2 among $1, 2, \ldots, n$.]