

4 Irreducible Components

Let K be an algebraically closed field.

Recall that $C \subseteq \mathbb{A}_2(K)$ is a *curve* if there is a polynomial $f \in K[X, Y]$ such that

$$C = V(f) = \{(x, y) \in \mathbb{A}_2(K) : f(x, y) = 0\}.$$

Definition 4.1 We say that a curve $C \subseteq \mathbb{A}_2(k)$ is *reducible* if there are curves $C_1, C_2 \subseteq C$ such that $C_1, C_2 \neq C$ and $C = C_1 \cup C_2$. Otherwise we say that C is *irreducible*.

For example the curve $X^2 - Y^2 = 0$ is reducible since

$$X^2 - Y^2 = (X - Y)(X + Y)$$

and $V(f)$ is the union of the two lines $X = Y$ and $X = -Y$. In general, if $f = \prod_{i=1}^n g_i$, then $V(f) = V(g_1) \cup \dots \cup V(g_n)$. But the $V(g_i)$ may not be distinct. For example $V(X^2) = V(X) \cup V(X) = V(X)$.

One of our goals of this section is to prove that any curve can be written as a finite union of irreducible curves in, essentially, a unique way. Clearly, to fully understand how to decompose a curve into irreducible components, we will need to understand factoring in $K[X, Y]$.

Unique Factorization Domains

We will prove that if k is a field, then the n -variable polynomial ring $k[X_1, \dots, X_n]$ is a unique factorization domain.

We begin by recalling some of basic definitions and results from Math 330.

Definition 4.2 Let D be an integral domain.

We say that $u \in D$ is a *unit* if there is $v \in D$ such that $uv = 1$. We write $v = \frac{1}{u}$.

If $a \in D$ is nonzero and not a unit, we say that a is *irreducible* if whenever $b, c \in D$ and $a = bc$ then either a is a unit or b is a unit.

We say that $a, b \in D$ are *associates* if there is a unit u such that $a = ub$.

For example, if k is a field and $D = k[X]$, then the units are the nonzero elements of k , $f \in k[X]$ is irreducible if and only if $\deg f > 0$ and f can not be factored as the product of two polynomials of lower degree, and f, g are associates if and only if there is $c \in k$ such that $f = cg$.

Lemma 4.3 *If D is an integral domain and $p \in D$ is irreducible, then $(p) = \{a \in D : p \text{ divides } a\}$ is a prime ideal (i.e., if $a, b \in D$ and p divides ab then p divides a or p divides b).*

Definition 4.4 If D is a domain, we say that D is a *Unique Factorization Domain* (or UFD) if:

i) if $f \in D$ is nonzero and not a unit, then there are irreducible elements $g_1, \dots, g_n \in D$ such that $f = g_1 g_2 \cdots g_n$, and

ii) if $p_1, \dots, p_n, q_1, \dots, q_m \in D$ are irreducible, and $p_1 \cdots p_n = q_1 \cdots q_m$, then $n = m$ and there is $\sigma \in S_n$ such that p_i is an associate of $q_{\sigma(i)}$ for $i = 1, \dots, n$. In other words, we can reorder the q_i so that p_i and q_i are associates for all i .

There are two important examples of UFDs that you have encountered before.

Theorem 4.5 *i) \mathbb{Z} is a UFD.*

ii) If k is a field, then $k[X]$ is a UFD.

The following theorem will allow us to construct more complicated UFDs.

Theorem 4.6 *If D is a UFD, then the polynomial ring $D[X]$ is a UFD.*

We can always identify the polynomial ring $D[X_1, \dots, X_n]$ with $D[X_1, \dots, X_{n-1}][X_n]$. Thus if D is a UFD, the polynomial ring in n -variables over D is as well.

Corollary 4.7 *i) $\mathbb{Z}[X_1, \dots, X_n]$ is a UFD.*

ii) If k is a field, then $k[X_1, \dots, X_n]$ is a UFD.

Definition 4.8 A nonconstant polynomial $f(X) = \sum_{n=0}^d a_n X^n \in D[X]$ is *primitive* if the only common divisors of a_0, \dots, a_n are units.

For example, let $D = k[X]$ and consider polynomials in $D[Y] = k[X, Y]$. The polynomial

$$X^2 Y + 2XY^2$$

is not primitive since X is a common nonunit divisor of X^2 and $2X^2$. The polynomial

$$X^2 + 2XY + (X + 1)Y^2$$

is primitive since $X^2, 2X$ and $(X + 1)$ have no common nonunit divisors. The polynomial $2X^2 Y + 4Y^2$ is primitive since the only common divisors of $2X^2$ and 4 are units.

If $f = \sum_{n=0}^d a_n X^n$ is not primitive, we can find a nonunit c such that for each n , $a_n = cb_n$ for some $b_n \in D$. Thus

$$f = c \sum_{n=0}^d b_n X^n.$$

Since c is a nonunit, f is reducible.

Note that $2X + 4Y$ is primitive in $\mathbb{Q}[X][Y]$, but not primitive in $\mathbb{Z}[X, Y]$, since 2 is a unit in \mathbb{Q} but not in \mathbb{Z} .

Proposition 4.9 *If $a \in D$ and a divides a nonconstant $g \in D[X]$, then a divides all of the coefficients of D . In particular, if g is primitive and $a \in D$ is a nonunit, then a does not divide g .*

Proof Suppose

$$g = a \left(\sum_{n=0}^d b_n X^n \right) = \sum_{n=0}^d ab_n X^n.$$

Then a divides each coefficient of g .

Lemma 4.10 *For any nonconstant $f \in D[X]$ we have $f(X) = cg(X)$ where $c \in D$ and $g(X)$ is primitive. Moreover if we also have $f(X) = dh(X)$, where $d \in D$ and h is primitive, then there is a unit $u \in D$ such that $h = gu$ and $d = \frac{c}{u}$. Thus c and g are “unique up to units”.*

We call c the content of g .

Proof

(existence) Suppose

$$f = \sum_{n=0}^d a_n X^n.$$

Since D is a unique factorization domain, we can factor each nonzero nonunit a_n into a product of irreducibles. Let M_f be the maximal number of irreducible factors occurring in the factorization of some a_n . We prove the lemma by induction on M_f .

If $M_f = 0$, then f is primitive.

If f is primitive, then we are done. Suppose f is not primitive. Then a_0, \dots, a_n have a common irreducible factor p . Let

$$h = \sum_{n=0}^d \frac{a_n}{p} X^n.$$

Since $M_h = M_f - 1$, by induction, there is $c \in D$ and a primitive g such that $h = cg$. But then $f = (pc)g$.

(uniqueness) Suppose $f = cg = dh$ where $c, d \in D$ and g, h are primitive. We claim that $h = ug$ for some unit u . We can factor c as the product of M_c irreducible factors. We will prove this by induction on M_c .

Suppose $p \in D$ is an irreducible factor of c . Since h is primitive, p does not divide all of the coefficients of h . Since it divides all the coefficients of f it must divide d . Similarly, if $p \in D$ is an irreducible factor of d , then p divides c .

If $M_c = 0$, then c is a unit and the above argument shows that d is also a unit. Thus $h = \frac{c}{d}g$ and $\frac{c}{d}$ is a unit.

If $M_c > 0$, then c is not a unit. If p is an irreducible factor of c , then p is also an irreducible factor of d and $\frac{c}{p}g = \frac{d}{p}h$. By induction $h = ug$ for some unit u .

For example, suppose k is a field, $D = k[X]$ and $f = X^2Y + 2XY^2 \in D[Y]$, then $f = X(XY + 2Y^2)$. Since $XY + 2Y^2$ is primitive in $D[Y]$, X is the content of f in $D[Y]$.¹

¹Note that we could also consider $D^* = k[Y]$ and $f \in D^*[X]$. In that case $f = Y(X^2 + 2XY)$ so the content is Y .

Lemma 4.11 (Gauss' Lemma) *If D is a UFD and $f, g \in D[X]$ are primitive, then fg is primitive. Indeed, if f_1, \dots, f_m are primitive, then so is $\prod f_i$.*

Proof Let

$$f = \sum_{n=0}^d a_n X^n$$

and

$$g = \sum_{n=0}^{d_1} b_n X^n$$

be primitive in $D[X]$. Suppose fg is not primitive. There is an irreducible p such that p divides every coefficient of fg . Let s be least such that p does not divide a_s and let t be least such that p does not divide b_t . The X^{s+t} -coefficient of fg is

$$\sum_{i+j=s+t} a_i b_j = \sum_{i<s} a_i b_{s+t-i} + a_s b_t + \sum_{j<t} a_{s+t-j} b_j.$$

By choice of s and t , p divides each a_i in the first sum on the right-hand side and each b_j in the second. Since p divides the whole sum, it must also divide $a_s b_t$. Thus p divides a_s or b_t , a contradiction.

The second claim is now a simple induction.

Exercise 4.12 Suppose D is a UFD and $f, g \in D[X]$. Prove that the content of f times the content of g is (a unit times) the content of fg .

Suppose D is a unique factorization domain. Let

$$K = \left\{ \frac{a}{b} : a, b \in D, b \neq 0 \right\}$$

be the fraction field of D . Then $D[X]$ is a subring of $K[X]$. Of course $F[X]$ is a UFD. We want to consider the relationship between factoring in $D[X]$ and factoring in $K[X]$. The first thing to ask is which elements of $D[X]$ are irreducible in $K[X]$.

Lemma 4.13 *Suppose D is a UFD, K is its fraction field and $f \in D[X]$. Then f is irreducible in $D[X]$ if and only if f is primitive and f is irreducible in $K[X]$.*

Proof

(\Leftarrow) Suppose there are nonunit $g, h \in D[X]$ such $f = gh$. If $g \in D$ or $h \in D$, then f is not primitive. If neither $g, h \in D$ then $\deg f > \deg g, \deg h$. Thus f is reducible in $K[X]$.

(\Rightarrow) We argued above that every irreducible $f \in D[X]$ is primitive, so we need only show it is irreducible in $K[X]$.

Suppose $f = gh$ where $\deg g, \deg h < \deg f$. We must show that f factors in $D[X]$. Each coefficient of g and h is a quotient of elements of D , by clearing denominators we can find $a \in D, g_1, h_1 \in D[X]$ such that $af = g_1 h_1$ and

$\deg f > \deg g_1, \deg h_1$. By Lemma 4.10 we can find $b, c, d \in D$ and primitive polynomials f_1, g_2, h_2 such that

$$f = bf_1, g_1 = cg_2, \text{ and } h_1 = dh_2.$$

Then

$$abf_1 = cdg_2h_2$$

and, by Gauss' Lemma g_2h_2 is primitive. Thus, by Lemma 4.10, there is a unit u , such that

$$abu = cd.$$

But then $bu = \frac{cd}{a}$ and

$$f = bf_1 = \frac{cd}{a}g_2h_2 = bug_2h_2.$$

Thus f factors into $(bug_2)h_2$ in $D[X]$.

Proof of Theorem 4.6

(existence) Suppose D is a UFD. We want show that $D[X]$ is a UFD. We first prove that every nonzero nonunit $f \in D[X]$ can be factored into a product of irreducibles. If $\deg f = 0$, then $f \in D$ and we are done since D is a UFD.

Suppose $\deg f > 0$. We may view f as an element of $K[X]$, where K is the fraction field of D . Since $K[X]$ is a UFD, we can factor

$$f = \prod_{i=1}^n g_i$$

where g_1, \dots, g_n are irreducible factors in $K[X]$. Let $a_i \in D$ be the product of all of the denominators occuring in coefficients in g_i , let $a = \prod a_i$ and let $h_i = a_i g_i \in D[X]$. Then

$$af = \prod h_i.$$

Since $a_i \in D$ is a unit in K , $h_i = a_i g_i$ is irreducible in $K[X]$. By Lemma 4.10, there are $c, d_1, \dots, d_n \in D$ and primitive $f^*, h_1^*, \dots, h_n^* \in D[X]$ such that

$$f = cf^*, h_1 = d_1 h_1^*, \dots, h_n = d_n h_n^*.$$

Note that each h_i^* is irreducible in $K[X]$ and primitive in $D[X]$. Thus by Lemma 4.13, each h_i^* is irreducible in $D[X]$.

But

$$acf^* = \prod_{i=1}^n d_i \prod_{i=1}^n h_i^*$$

and, by Gauss' Lemma, $\prod h_i^*$ is primitive. Thus, by 4.10 there is a unit $u \in D$ such that

$$acu = \prod_{i=1}^n d_i.$$

Then

$$acf^* = acu \prod_{i=1}^n h_i^*$$

and

$$f = cf^* = cu \prod_{i=1}^n h_i^*.$$

Since D is a UFD, we can factor cu into a product of irreducibles. This gives a factorization of f into a product of irreducibles.

(uniqueness) If $\deg f = 0$, then $f \in D$ and we have unique factorization since D is a UFD. Suppose $\deg f > 0$. Suppose

$$f = a_1 \cdots a_s p_1 \cdots p_m = b_1 \cdots b_t q_1 \cdots q_n$$

are two irreducible factorizations of $f \in D[X]$ where a_i, b_j are irreducible elements of D and p_i, q_j are irreducible nonconstant elements of $D[X]$. By Lemma 4.13, each p_i and q_j is irreducible in $K[X]$. In $K[X]$, $\prod p_i$ is a unit times $\prod q_i$. Thus, since $K[X]$ is a UFD, $m = n$ and we can reorder the q_i such that $p_i = \frac{c_i}{d_i} q_i$ for some $c_i, d_i \in D$. But then $d_i p_i = c_i q_i$ and each p_i and q_j is irreducible in $D[X]$ and, hence, primitive. Thus $\frac{c_i}{d_i}$ is a unit in D .

Thus there is a unit $u \in D$ such that

$$\prod_{i=1}^s a_i \prod_{i=1}^m p_i = u \prod_{i=1}^n b_i \prod_{i=1}^m p_i.$$

Since D is a UFD, $s = t$ and we can reorder the b_i such that $a_i = w_i b_i$ for some unit. Thus the factorization is unique up to permuting the factors and multiplication by units.

Study's Lemma

We return to trying to understand algebraic curves in $\mathbb{A}_2(K)$.

For the remainder of this section K is an algebraically closed field

Our first goal is to relate divisibility in $K[X, Y]$ with containment of curves. If $f, g \in K[X, Y]$ and f divides g , then there is $h \in K[X, Y]$ such that $g = fh$ and $V(g) = V(f) \cup V(h)$. Thus $V(f) \subseteq V(g)$. We will argue that the converse is also true for irreducible f .

Theorem 4.14 (Study's Lemma) *If $f, g \in K[X, Y]$, f is irreducible and nonconstant, and $V(f) \subseteq V(g)$, then f divides g .*

Note that Study's Lemma may fail if K is not algebraically closed. For example $X^2 + Y^2$ is irreducible in $\mathbb{R}[X, Y]$ and $V(X^2 + Y^2) \subset V(X)$ in $\mathbb{A}_2(\mathbb{R})$, but $X^2 + Y^2$ does not divide X .

Before proving Study's Lemma, we derive some consequences.

Corollary 4.15 *If $f \in K[X, Y]$ is nonconstant, then $V(f) \neq \emptyset$.*

Proof If $V(f) = \emptyset \subseteq V(1)$, then, by Study's Lemma, f divides 1 and f is constant.

Exercise 4.16 Show that $V(f)$ is infinite.

Corollary 4.17 Let $f \in K[X, Y]$ be nonconstant. The curve $V(f) \subseteq \mathbb{A}_2(K)$ is irreducible, if and only if there is an irreducible polynomial g and $k > 0$ such that $f = g^k$.

Proof

(\Rightarrow) Since $K[X, Y]$ is a UFD, we can factor $f = \prod_{i=1}^n g_i^{k_i}$ where g_1, \dots, g_n are irreducible and relatively prime. If $n > 1$, then $V(f) = \bigcup V(g_i)$ and $V(g_i) \neq V(f)$, contradicting the irreducibility of f .

(\Leftarrow) Suppose $f = g^k$ where g is irreducible and $V(f) = V(h_1) \cup V(h_2)$ where $V(h_1) \neq V(h_2)$. Let p be an irreducible factor of h_1 . Then $V(p) \subseteq V(h_1) \subset V(f)$. By Study's Lemma, p divides g^k . Since g is irreducible $p = cg$ for some $c \in K$ and $V(p) = V(g) = V(f)$.

Theorem 4.18 If $f \in K[X, Y]$ is nonconstant and $C = V(f)$, then there are irreducible curves C_1, \dots, C_n such that $C = C_1 \cup \dots \cup C_n$ and $C_i \not\subseteq C_j$ for $i \neq j$. Moreover, if $D \subseteq C$ is an irreducible curve, then $D = C_i$ for some i .

We call C_1, \dots, C_n the irreducible components of C .

Proof We can factor

$$f = \prod_{i=1}^n g_i^{k_i}$$

where each g_i is irreducible and g_i and g_j are relatively prime for $i \neq j$. Then

$$C = V(g_1) \cup \dots \cup V(g_n)$$

and, by Corollary 4.17, each $V(g_i)$ is irreducible. By Study's Lemma, $C_i \not\subseteq C_j$ for $i \neq j$.

If $D \subseteq C$ is an irreducible curve, there is an irreducible $h \in K[X, Y]$ such that $D = V(h)$. Since $V(h) \subseteq V(f)$, h divides f . Since $K[X, Y]$ is a unique factorization domain, $h = cg_i$ for some $i \leq n$ and $c \in K$. Thus $V(h) = V(g_i)$.

We still must prove Study's Lemma. The proof will use a generalization of Theorem 1.22 on resultants. Suppose D is an integral domain and $f, g \in D[X]$ are nonzero polynomials. If $f = a_n X^n + a_{n-1} X^{n-1} + \dots + a_0$ and $g = b_m X^m + b_{m-1} X^{m-1} + \dots + b_0$ where $a_n, b_m \neq 0$, we can still form the resultant

$$R_{f,g} = \begin{vmatrix} a_0 & a_1 & \dots & \dots & \dots & a_n & 0 & \dots & \dots & 0 \\ 0 & a_0 & a_1 & \dots & \dots & \dots & a_n & 0 & \dots & 0 \\ & & & \ddots & \ddots & & & & & \\ 0 & \dots & \dots & 0 & a_0 & a_1 & \dots & \dots & \dots & a_n \\ b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & \dots & \dots & 0 \\ 0 & b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & \dots & 0 \\ & & & \ddots & \ddots & & & & & \\ 0 & \dots & \dots & \dots & 0 & b_0 & b_1 & \dots & \dots & b_m \end{vmatrix}.$$

We will have $R_{f,g} \in D$.

We need a mild generalization of Theorem 1.22

Theorem 4.19 *Suppose D is a unique factorization domain and $f, g \in D[X]$ are nonzero polynomials. The following are equivalent:*

- i) f and g have a common nonconstant factor in $D[X]$;*
- ii) There are nonzero $f_1, g_1 \in D[X]$ such that $\deg f_1 < \deg f$, $\deg g_1 < \deg g$ and*

$$fg_1 = f_1g.$$

- iii) $R_{f,g} = 0$.*

Exercise 4.20 Show that the proof of Theorem 1.22 can be modified to prove Theorem 4.19. [Note: The proof of ii) \Leftrightarrow iii) should work for any integral domain D .]

Proof of Study's Lemma

We have $f, g \in K[X, Y]$, f is irreducible and $V(f) \subseteq V(g)$. Without loss of generality assume that Y occurs in some monomial of f (otherwise we work with X instead of Y). We can write

$$f = \sum_{i=0}^n a_i Y^i$$

and

$$g = \sum_{i=0}^m b_i Y^i$$

where $a_i, b_i \in K[X]$ for all i , and $a_n, b_m \neq 0$.

Suppose for purposes of contradiction, that f does not divide g . Since f is irreducible, f and g have no common nonconstant factors. Thus by Theorem 1.22, $R_{f,g} \in K[X]$ is nonzero. Since K is algebraically closed K is infinite. The polynomials $a_n, b_m, R_{f,g} \in K[X]$ are nonzero, thus we can find $c \in K$ such that $a_n(c) \neq 0$, $b_m(c) \neq 0$ and $R_{f,g}(c) \neq 0$.

Consider the polynomials

$$p(Y) = f(c, Y) = \sum_{i=0}^n a_i(c) Y^i$$

and

$$q(Y) = g(c, Y) = \sum_{i=0}^m b_i(c) Y^i.$$

Note that p has degree $n > 0$, q has degree m and

$$R_{p,q} = \begin{vmatrix} a_0(c) & a_1(c) & \dots & \dots & \dots & a_n(c) & 0 & \dots & \dots & 0 \\ 0 & a_0(c) & a_1(c) & \dots & \dots & \dots & a_n(c) & 0 & \dots & 0 \\ & & & & \ddots & \ddots & & & & \\ 0 & \dots & \dots & 0 & a_0(c) & a_1(c) & \dots & \dots & \dots & a_n(c) \\ b_0(c) & b_1(c) & \dots & \dots & b_m(c) & 0 & \dots & \dots & \dots & 0 \\ 0 & b_0(c) & b_1(c) & \dots & \dots & b_m(c) & 0 & \dots & \dots & 0 \\ & & & & \ddots & \ddots & & & & \\ 0 & \dots & \dots & \dots & 0 & b_0(c) & b_1(c) & \dots & \dots & b_m(c) \end{vmatrix}$$

Thus $R_{p,q} = R_{f,g}(c) \neq 0$. Since $\deg p > 0$ there is $d \in K$ such that $p(d) = 0$ and, by Theorem 1.22, $q(d) \neq 0$. But then $f(c, d) = 0$ and $g(c, d) \neq 0$ contradicting the fact that $V(f) \subseteq V(g)$.

Irreducible Components in Projective Space

We next consider irreducible components in $\mathbb{P}_2(K)$. If $F \in K[X, Y, Z]$ is homogeneous, then, because $K[X, Y, Z]$ is a UFD, we can factor F into irreducible factors. We first need to notice that the irreducible factors are homogeneous.

Lemma 4.21 *Suppose $F, G, H \in K[X_1, \dots, X_n]$, F is homogeneous and $F = GH$. Then G and H are homogeneous.*

Proof We can write $G = \sum_{i=0}^n G_i$ and $H = \sum_{i=0}^m H_i$ where G_i and H_i are homogeneous of degree i , $G_n \neq 0$ and $H_m \neq 0$. Let i_0 be least such that $G_{i_0} \neq 0$ and let j_0 be least such that $H_{j_0} \neq 0$. Then

$$F = G_n H_m + \sum_{k=i_0+j_0+1}^{m+n-1} \sum_{i+j=k} G_i H_j + G_{i_0} H_{j_0}$$

and

$$\sum_{i+j=k} G_i H_j$$

is homogeneous of degree k . Since F is homogeneous we must have $n = i_0$ and $m = j_0$. Thus $G = G_n$ and $H = H_m$ are homogeneous.

Our next goal is to prove the projective version of Study's Lemma. We must first investigate the relationship between factoring polynomials in $K[X, Y]$ and factoring their homogenizations in $K[X, Y, Z]$.

Recall that if $f \in K[X, Y]$ has degree d . We can write $f = \sum_{i=0}^d f_i$ where $f_i \in K[X, Y]$ is homogeneous of degree i and let

$$F(X, Y, Z) = \sum_{i=0}^d f_i Z^{d-i}.$$

Then F is homogeneous of degree d and $f(X, Y) = F(X, Y, 1)$. We call F the *homogenization* of f . Note that Z is not a factor of the homogenization.

Lemma 4.22 Suppose $f, g, h \in K[X, Y]$ are nonconstant polynomials and $f = gh$. Suppose F, G, H in $K[X, Y, Z]$ are the homogenizations of f, g and h . Then $F = GH$.

Proof Let

$$f = \sum_{i=0}^{n+m} f_i, \quad g = \sum_{i=0}^n g_i, \quad \text{and} \quad h = \sum_{i=0}^m h_i$$

where f_i, g_i, h_i are homogeneous of degree i , $g_n, h_m, f_{n+m} \neq 0$. Then

$$f_k = \sum_{i+j=k} g_i h_j.$$

But

$$G = \sum_{i=0}^n g_i Z^{n-1}, \quad H = \sum_{i=0}^m h_i Z^{m-i}$$

and

$$GH = \sum_{k=0}^{m+n} \left(\sum_{i+j=k} g_i Z^{n-i} h_j Z^{m-j} \right) = \sum_{k=0}^{m+n} f_k Z^{m+n-k} = F.$$

Corollary 4.23 Suppose $f \in K[X, Y]$ and $F \in K[X, Y, Z]$ is its homogenization. Then f is irreducible in $K[X, Y]$ if and only if F is irreducible in $K[X, Y, Z]$.

Proof

(\Leftarrow) Clear from the Lemma.

(\Rightarrow) Suppose $F = GH$. If $g(X, Y) = G(X, Y, 1)$ and $h(X, Y) = H(X, Y, 1)$, then $f = gh$. Since f is irreducible, one of g or h is constant. Suppose g is constant. Since Z is not a factor of F , Z is not a factor of G . Thus $\deg g = \deg G$ and G is constant. Thus F is irreducible.

We can now prove the projective version of Study's Lemma. If $F \in K[X, Y, Z]$ is homogeneous, we let

$$V_{\mathbb{P}}(F) = \{p \in \mathbb{P}_2(K) : F(p) = 0\}.$$

Proposition 4.24 Suppose K is an algebraically closed field. If $F, G \in K[X, Y, Z]$ are homogeneous, F is irreducible and $V_{\mathbb{P}}(F) \subseteq V_{\mathbb{P}}(G)$, then F divides G .

Proof There are two cases to consider.

case 1: $F = aZ$ for some $a \in K \setminus \{0\}$.

We will suppose F does not divide G and show that $V_{\mathbb{P}}(F) \not\subseteq V_{\mathbb{P}}(G)$. Let

$$G(X, Y) = \sum_{i=0}^n g_i(X, Y) Z^i$$

where $g_i \in K[X, Y]$ is homogeneous of degree $n - i$. Since Z does not divide G , $g_0 \neq 0$. Thus there are $x, y \in K$ such that $x \neq 0$, $y \neq 0$ and $g_0(x, y) \neq 0$. But then $[x, y, 0] \in V_{\mathbb{P}}(F) \setminus V_{\mathbb{P}}(G)$ and $V_{\mathbb{P}}(F) \not\subseteq V_{\mathbb{P}}(G)$.

case 2: Z does not divide F .

We can factor $G = Z^d H$ where Z does not divide H . Let $f(X, Y) = F(X, Y, 1)$, and $h(X, Y) = H(X, Y, 1)$. If $f(x, y) = 0$, then $h(x, y) = 0$. Thus $V(f) \subseteq V(h)$ and by Study's Lemma, f divides h . By Lemma 4.22, F divides H . Thus F divides G .

We can now follow the arguments given above for affine curves. We say that a projective curve C is irreducible, if there are no projective curves $D_0, D_1 \subset C$ with $C = D_0 \cup D_1$, $C \neq D_0$, and $C \neq D_1$.

Exercise 4.25 Suppose K is an algebraically closed field and $F \in K[X, Y, Z]$ is nonconstant and homogeneous.

a) Show that $V_{\mathbb{P}}(F)$ is irreducible if and only if $F = G^k$ for some irreducible $G \in K[X, Y, Z]$.

b) Let C be a projective curve. Show that there are irreducible projective curves C_1, \dots, C_n such that $C = C_1 \cup \dots \cup C_n$ and $C_i \not\subseteq C_j$ for $i \neq j$. Moreover, if $D \subseteq C$ is an irreducible curve, then $D = C_i$ for some i . We call C_1, \dots, C_n the irreducible components of C .

c) Suppose $f = F(X, Y, 1)$ and Z does not divide F . Show that if C_1, \dots, C_n are the irreducible components of $V_{\mathbb{P}}(F)$, then $C_1 \cap \mathbb{A}_2(K), \dots, C_n \cap \mathbb{A}_2(K)$ are the irreducible components of the affine curve $V(f)$.

d) What happens in c) if Z divides F ?