Math 494: Topics in Algebra
Problem Set 2

Due Wednesday March 5: Turn in five of the following problems.

1) Let $AG_n(k)$ be the set of all invertible affine transformations of $\mathbb{A}_n(k)$.

a) Prove that $AG_n(k)$ is a group under composition.

b) We say that $T \in AG_n(k)$ is a *translation* if there is $\vec{b}$ such that

$$T(\vec{x}) = \vec{x} + \vec{b}$$

for all $\vec{x} \in \mathbb{A}_n(k)$. Let N be the set of all translations of $\mathbb{A}_n(k)$. Prove that N is a normal subgroup of $AG_n(k)$.

c) Prove that $AG_n(k)/N$ is isomorphic to $\mathrm{GL}_n(k)$.

d) Prove that $AG_1(k)$ is isomorphic to the group of matricies

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in k, a \neq 0 \right\}.$$

2) Let $PG_n(k)$ be the set of all projective transformations of $\mathbb{P}_n(k)$. For $A \in \mathrm{GL}_{n+1}(k)$ we let $T_A : \mathbb{P}_n(k) \to \mathbb{P}_n(k)$ be the transformation

$$T_A([\vec{x}]) = [A\vec{x}].$$

a) Show that $PG_n(k)$ is a group under composition.

b) Show that $A \mapsto T_A$ is a homomorphism from $\mathrm{GL}_{n+1}(k)$ to $PG_n(k)$.

c) What is the kernel of this homomorphism?

d) Recall that $\mathrm{SL}_n(k) = \{A \in k : \det\,(A) = 1\}$. Let I be the identity matix. Prove that $PG_2(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$.

3) a) Suppose $a \neq b \in k$. Prove that there is a unique $T \in AG_1(k)$ with $T(0) = a$ and $T(1) = b$. Conclude that for any $x \neq y$ and any $a \neq b$ there is a unique $T \in AG_1(k)$ with $T(x) = a$ and $T(y) = b$.

b) Suppose $a, b, c \in \mathbb{P}_1(k)$ are distinct. Prove that there is a unique T in $PG_1(k)$ such that $T(0) = a$, $T(1) = b$ and $T(\infty) = c$. Conclude that for any distinct $x, y, z \in \mathbb{P}_1(k)$, and any distinct $a, b, c \in \mathbb{P}_1(k)$ there is a unique $T \in PG_1(k)$ with $T(x) = a$, $T(y) = b$ and $T(z) = c$.

4) Let K be an algebraically closed field of characteristic 2 and let C be the curve $X^2 + Y^2 = 1$. Show that C is a line.

This shows one reason why when we study conics we exclude the characteristic 2 case.

5) Give a classification of conics in $\mathbb{P}_2(\mathbb{R})$ similar to the one given in Corollary 3.19.

6) Let C be the ellipse $X^2 + 2Y^2 = 6$.

a) Find rational functions $f(t), g(t) \in \mathbb{Q}(t)$ such that $(f(t), g(t)) \in C$ for all $t \in \mathbb{R}$.

b) Find all $(a, b, c) \in \mathbb{Z}^3$ such that $a^2 + 2b^2 = 6c^2$.