Math 494: Topics in Algebra
Problem Set 4

Due Friday April 4: Do all four of the following problems.

We assume throughout that K is an algebraically closed field.

1) Let C be an affine curve. Suppose $f \in K[X,Y]$ is of minimal degree such that $C = V(f)$. We call f a *minimal polynomial* for C.

a) Show that if p is irreducible and p divides f, then p^2 does not divide f.

b) Show that if g is another minimal polynomial for f, then $g = af$ for some nonzero $a \in K$.

2) Let

$$f(X,Y) = X^3 - 3X^2Y + 2X^2 + 3XY - 9Y^2 + 9Y - X - 2$$

$$g(X,Y) = 2X^3 - X^2Y + X^2 + 6XY - 3Y^2 + 4Y - 2X - 1$$

and

$$h(X,Y) = X^2Y^2 - 2X^4 + X^2Y - X^2 + 2XY^3 - 4X^3Y + 2XY^2 - 2XY.$$

a) Do f and g have a common nonconstant factor?

b) Do f and h have a common nonconstant factor? [Hint: Use resultants in MAPLE.]

3) Suppose $F \in K[X,Y,Z]$ is nonconstant and homogeneous.

a) Show that $V_{\mathbb{P}}(F)$ is irreducible if and only if $F = G^k$ for some irreducible $G \in K[X,Y,Z]$.

b) Let C be a projective curve. Show that there are irreducible projective curves $C_1, \ldots, C_n$ such that $C = C_1 \cup \ldots \cup C_n$ and $C_i \not\subseteq C_j$ for $i \neq j$. Moreover, if $D \subseteq C$ is an irreducible curve, then $D = C_i$ for some i. We call $C_1, \ldots, C_n$ the irreducible components of C.

4) Suppose $F \in K[X,Y,Z]$ is a nonconstant homogenous polynomial.

a) Suppose $f = F(X,Y,1)$ and Z does not divide F. Show that if $C_1, \ldots, C_n$ are the irreducible components of $V_{\mathbb{P}}(F)$, then $C_1 \cap \mathbb{A}_2(K), \ldots, C_n \cap \mathbb{A}_2(K)$ are the irreducible components of the affine curve $V(f)$.

b) What happens in a) if Z divides F?

5) Suppose $F \in K[X,Y,Z]$ is nonzero. Consider the polynomial $F(TX,TY,TZ) \in K[X,Y,Z,T]$. Then F is homogeneous of degree d if and only if

$$F(TX,TY,TZ) = T^d F(X,Y,Z).$$