

Math 494: Topics in Algebra
Problem Set 5

Due 3:00pm Tuesday May 6

1) Let

$$\begin{aligned} F &= ZY + Z^2 - X^2 \\ G_1 &= 3ZY + 2Z^2 + Y^2 - X^2 \\ G_2 &= -3ZY - Z^2 + YX + ZX - X^2 - 2Y^2 \\ G_3 &= X^2 + 2Y^2 - 2Z^2 \end{aligned}$$

a) For $i = 1, 2, 3$ find the point of intersection of $V_{\mathbb{P}}(F)$ and $V_{\mathbb{P}}(G_i)$ and their multiplicities. (Hint: $|V_{\mathbb{P}}(F) \cap V_{\mathbb{P}}(G_i)| = i$.)

b) Find irreducible conics C and D such that $|C \cap D| = 4$. Thus if C and D are conics $|C \cap D|$ can be 1,2,3 or 4.

2) Let $F = (X^2 + Y^2)^2 + 3X^2YZ - Y^3Z$.

a) Show that $[0, 0, 1]$ is a singular point of $V_{\mathbb{P}}(F)$ of order 3.

b) What are the tangent lines at $[0, 0, 1]$?

c) Find all points of $V_{\mathbb{P}}(F)$ on the line $Z = 0$. Find the tangent line at each one.

d) Find a rational parameterization of $V_{\mathbb{P}}(F)$.

3) Let $F = 4X^3 - aXZ^2 - bZ - Y^2Z$.

a) Show that $V_{\mathbb{P}}(F)$ is smooth if and only if $a^3 \neq 27b^2$.

b) Show that $[0, 1, 0]$ is a point of inflection of $V_{\mathbb{P}}(F)$.

4) We say that a curve C is a *folium of Descartes* if it is $V_{\mathbb{P}}(F)$ where $F = X^3 + Y^3 - 3aXYZ$ for some $a \neq 0$. Show that if C is a folium of Descartes, then so is its Hessian curve $V_{\mathbb{P}}(\det(H_F))$.

5) Let C be a curve of degree d and L is a line. Suppose that p_1, \dots, p_m are distinct singular points of C such that $p_1, \dots, p_n \in L$. Prove that if $2m > d$ then $L \subset C$.