Math 502 Metamathematics I
Problem Set 1

Due: Friday September 12

1) Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure, $\sigma : V \to M$ is an assignment, ϕ is an $\mathcal{L}$-formula and $M \models_\sigma \exists v_1 \forall v_2\ \phi$. Prove that $\mathcal{M} \models_\sigma \forall v_2 \exists v_1\ \phi$.

2)[1] a) Suppose that $\phi_1, \ldots, \phi_n$ are $\mathcal{L}$-formulas and ψ is a Boolean combination of $\phi_1, \ldots, \phi_n$ (i.e., ψ is built from $\phi_1, \ldots, \phi_n$ using only $\wedge$, $\vee$ and $\neg$). Prove there is $S \subseteq \mathcal{P}(\{1, \ldots, n\})$ such that

$$\models \psi \Leftrightarrow \bigvee_{X \in S} \left(\bigwedge_{i \in X} \phi_i \wedge \bigwedge_{i \notin X} \neg \phi_i \right).$$

b) Show that every formula is equivalent to one of the form

$$Q_1 v_1 \ldots Q_m v_m\ \psi,$$

where ψ is quantifier-free and each Q_i is either $\forall$ or $\exists$.

3) Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups. Show that there is a sentence ϕ such that $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. [Note: More generally, if $\mathcal{L}$ is any finite language and $\mathcal{M}$ is a finite $\mathcal{L}$-structure. There is an $\mathcal{L}$-sentence ϕ such that $\mathcal{N} \models \phi$ if and only if $\mathcal{N} \cong \mathcal{M}$.]

4) Let $\mathcal{L}$ be any countable language. Show that for any infinite cardinal κ there are at most 2^κ nonisomorphic $\mathcal{L}$-structures of cardinality κ.

---

[1]Some notation:
- $\mathcal{P}(A)$ is the set of all subsets of A.
- If S is the finite set $\{\phi_1, \ldots, \phi_m\}$ then

$$\bigvee_{\phi \in S} \phi \text{ and } \bigwedge_{\phi \in S} \phi$$

are the formulas $\phi_1 \vee \ldots \phi_m$ and $\phi_1 \wedge \ldots \wedge \phi_m$ respectively.