Math 502 Metamathematics I
Problem Set 2

Due: Friday September 26

Problems marker with a † are more difficult and optional.

1) Let T and T' be $\mathcal{L}$-theories with $T' \subseteq T$. We say that T' *axiomatizes* T if every model of T' is a model of T. Suppose T' axiomatizes T. Prove that $T \models \phi$ if and only if $T' \models \phi$ for all $\mathcal{L}$-sentences ϕ.

2) Let $\mathcal{L} = \{+, 0\}$. Prove that $\mathrm{Th}(\mathbb{Z}) \neq \mathrm{Th}(\mathbb{Z} \oplus \mathbb{Z})$.

3) Let $\mathcal{L} = \{R\}$. Let T_0 be the theory:
$\forall x \, \neg R(x, x)$ (R is irreflexive)
$\forall x \forall y \, (R(x, y) \to R(y, x))$ (R is symmetric)
T_0 is the theory of graphs. Show how to axiomatize the following classes:
a) complete graphs;
b) acyclic graphs;
c) graphs of valence 2 (i.e,. graphs where every element has an edge to exactly two other elements);
d)† bipartite graphs.

4) a) Let $\mathcal{L}$ be the language $\{+, 0\}$ and consider the structure $\mathcal{R}$ with universe $\mathbb{R}$ where $+$ is interpreted as the usual addition and 0 as zero. Show that there is no formula $\phi(v, w)$ such that $\mathcal{R} \models \phi(a, b)$ if and only if $a < b$ for all $a, b \in \mathbb{R}$. [**Hint:** Find a and b and an automorphism F of $\mathbb{R}$ such that $a < b$ but $F(a) > F(b)$.]

b)† Let $\mathcal{L}$ be the language of rings $\{+, \cdot, 0, 1\}$ and let $\mathcal{R}$ be $\mathbf{R}$ with the usual interpretation. Find a formula $\phi(v, w)$ such that $a < b$ iff $\mathcal{R} \models \phi(a, b)$.

5) If ϕ is a sentence, the *spectrum* of ϕ is the set of all natural numbers n such that there is a model of ϕ with exactly n elements.

a) Let $\mathcal{L} = \{E\}$ where E is a binary relation. Write down a sentence ϕ asserting that E is an equivalence relation and every equivalence class has exactly three elements. Show that the spectrum of ϕ is $\{n > 0\colon 3 \text{ divides } n\}$.

b) Let $\mathcal{L} = \{P, Q, f\}$ where P and Q are unary predicates and f is a binary function. Let ϕ be the conjuction of:
$\exists x \exists y \; x \neq y \wedge P(x) \wedge P(y)$
$\exists x \exists y \; x \neq y \wedge Q(x) \wedge Q(y)$
$\forall z \exists x \exists y \; P(x) \wedge Q(y) \wedge f(x,y) = z$
$\forall x_1 \forall x_2 \forall y_1 \forall y_2 \; [(P(x_1) \wedge P(x_2) \wedge Q(y_1) \wedge Q(y_2) \wedge f(x_1, y_1) = f(x_2, y_2)) \rightarrow$
$(x_1 = x_2 \wedge y_1 = y_2)]$
Show that the spectrum of $\phi = \{n > 3 : n \text{ is not prime}\}$.[1]

c) Find a sentence with the spectrum $\{n > 0 : n \text{ is a square }\}$.

d) Find a sentence with the specturm $\{p^n : p \text{ prime } n > 0\}$.

e)†† Find a sentence with spectrum $\{p : p \text{ is prime}\}$.

f)††† Prove that $X \subseteq \mathcal{N} \setminus \{0\}$ is a spectrum if and only if X is in recognizable in nondeterministic exponential time.[2]

[1]Another sentence with the same spectrum is the sentence in the language of rings asserting that we have a commutative ring **with** zero divisors

[2]The question about whether the complement of a spectrum is a spectrum is an old open question. By this result it is equivalent to the question about whether nondeteministic exponential time is co-nondeteministic exponential time.