Math 502 Metamathematics I
Problem Set 4

Due: Wednesday October 22

1) Let $\mathcal{L} = \{E\}$ where E is a binary relation. Let Γ_0 be the axioms for equivalence relations. Suppose $\Gamma \supseteq \Gamma_0$ is an $\mathcal{L}$-theory such that for all n there is $\mathcal{M} \models \Gamma$ with an equivalence class of size at least n. Then there is $\mathcal{M} \models \Gamma$ with an infinite equivalence class. Conclude that the class of all equivalence relations where every clas is finite is not an elementary class.

2) We say that an ordered fields $(F, +, \cdot, <)$ is *archimedian* if

for all $x, y \in F$ with $x, y > 0$ there is an integer m such that $x < my$.

Show that there is a non-archimedian field $F \models \mathrm{Th}(\mathbb{R})$. Note that F contains infinite and infinitesimal elements.

3) Let $\mathcal{L}$ be the language with one binary relation symbol $<$. Let T be an $\mathcal{L}$-theory extending the theory of linear orders such that T has infinite models. Show that there is $\mathcal{M} \models T$ and an order preserving embedding $\sigma : \mathbb{Q} \to M$ of the rational numbers into M. [Hint: Add constants c_q for all $q \in \mathbb{Q}$ and let $T^* = T \cup \{c_q < c_r : q, r \in \mathbb{Q}, q < r\}$.]

For example there is $\mathcal{M} \models \mathrm{Th}(\mathbb{Z}, <)$ in which the rational order embedds.

4)† Show that every torsion free abelian group $(G, +)$ can be linearly ordered such that $(a < b \wedge c \leq d) \to a + c < b + d$. [Hint: First show this for finitely generated groups. Then use compactness.]