Math 502 Metamathematics I
Problem Set 5

Due: Friday November 7: Do at least 4 of the following problems.

1) Let $\mathcal{L} = \{s\}$, where s is a unary function symbol. Let T be the $\mathcal{L}$-theory that asserts that s is a bijection with no cycles (i.e., $s^{(n)}(x) \neq x$ for $n = 1, 2, \ldots$). For which cardinals κ is T κ-categorical? Is T complete?

2) Let T be the theory of Abelian groups where every element has order 2. Show that T is κ-categorical for all infinite cardinals κ but not complete. Find $T' \supset T$ a complete theory with the same infinite models as T. [Hint: models of T are vector spaces over $\mathbb{F}_2$, the two element field.]

3) a) Prove that there is a dense linear order without endpoints $(A, <)$ (i.e. $(A, <) \models$ DLO with $|A| = |\mathbb{R}|$ and $(A, <) \not\cong (\mathbb{R}, <)$. This proves that DLO is not $2^{\aleph_0}$-categorical.

b)† Prove that DLO is not κ-categorical for any uncountable κ.

c)†† Prove that for any uncountable κ there are 2^κ nonisomorphic models of DLO of cardinality κ.

4) a) Write a register machine program to compute $\max(x, y)$.

b) Give primitive recursive functions $j(x, y, s)$ and $g_i(x, y, s)$ such that if the program from a) is given input x and y, $j(x, y, s)$ is the next instruction and $g_i(x, y, s)$ is the contents of register i at time s.

5) Write a register machine program to compute lcm(x, y), the least common multiple of x and y.

6) Prove that $\max(x, y)$ and lcm(x, y) are primitive recursive.

7) Suppose $P(\bar{x}, y)$ is a primitive recursive predicate and $g(\bar{x})$ is a primitive recursive function. Define $f(\bar{x}) = 0$ if there is no $n \leq g(\bar{x})$ such that $P(\bar{x}, n) = 1$. Otherwise $f(\bar{x})$ is the least $n \leq g(\bar{x})$ such that $P(\bar{x}, n) = 1$. Prove that f is primitive recursive.

Bonus Problem Exercise 6.13 in the notes.