Math 502 Mathematical Logic
Problem Set 1

Due: Friday September 4

Let $\mathcal{L}$ be a language with relation symbols $\mathcal{R}$, function symbols $\mathcal{F}$ and constant symbols $\mathcal{C}$.

1) Let $T_0 = \mathcal{C} \cup \{v_1, v_2, \ldots\}$ be the variables and constants of the language. Given T_k let

$$T_{k+1} = \{f(t_1, \ldots, t_{n_f}) : f \in \mathcal{F}, t_1, \ldots, t_{n_f} \in T_k\} \cup T_k.$$

Let $T = \bigcup_{k=0}^{\infty} T_k$. Prove that T is the set of all terms. [Note: We can give a similar construction of the set of all formulas]

2) a) Prove that for every term is finite. [Hint: Show that the set of finite terms contains all constants and variables and is closed under the term formation rule and use the fact that the set of terms is minimal with this property. Alternatively, you can use 1)]

b) Prove that every formula is finite.

3) Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $\sigma : V \to M$ be an assignment.

a) Suppose $\mathcal{M} \models_\sigma \exists v_1 \forall v_2\ \phi$. Prove that $\mathcal{M} \models_\sigma \forall v_2 \exists v_1\ \phi$. Give an example showing that the converse is false.

b) Prove that $\mathcal{M} \models_\sigma \forall\ v_1 \phi$ if and only if $\mathcal{M} \models_\sigma \neg \exists v_1\ \neg \phi$.

c) Give an example of $\mathcal{L}$, $\mathcal{M}$, σ, ϕ and ψ such that $\mathcal{M} \models_\sigma (\exists v_1\ \phi \wedge \psi)$ but $\mathcal{M} \not\models_\sigma \exists v_1\ (\phi \wedge \psi)$.

4) The set of *positive $\mathcal{L}$-formulas* is the smallest set of $\mathcal{L}$-formulas containing the atomic formulas such that:

i) if ϕ and ψ are positive formulas, then so are $(\phi \wedge \psi)$ and $(\phi \vee \psi)$;
ii) if ϕ is a positive formula, then so are $\exists v_i\ \phi$ and $\forall v_i\ \phi$.

Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures we say that $j : M \to \mathcal{N}$ is an $\mathcal{L}$-*homomorphism* if:

i) $j(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all $c \in \mathcal{C}$;
ii) $j(f^{\mathcal{M}}(\overline{a})) = f^{\mathcal{N}}(j(\overline{a}))$ for all $\overline{a} \in M$ and $f \in \mathcal{F}$;
iii) if $\overline{a} \in R^{\mathcal{M}}$, then $j(\overline{a}) \in R^{\mathcal{N}}$ for $R \in \mathcal{R}$.

Suppose $j : M \to N$ is a surjective homomrophism and $\sigma : V \to \mathcal{M}$ is an assignment,

a) Show that if t is an $\mathcal{L}$-term, then $j(t^{\mathcal{M}}[\sigma]) = t^{\mathcal{N}}[j \circ \sigma]$.
b) Suppose ϕ is a positive formula and $\mathcal{M} \models_\sigma \phi$. Prove that $\mathcal{N} \models_{j \circ \sigma} \phi$.