Math 502 Mathematical Logic
Problem Set 4

Due: Friday October 2

We say that $\mathcal{K}$ is an *elementary class* if there is a theory T such that $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$.
1) Let $\mathcal{L} = \{E\}$ where E is a binary relation. Let T_0 be the axioms for equivalence relations. Suppose $T \supseteq T_0$ is an $\mathcal{L}$-theory such that for all n there is $\mathcal{M} \models T$ with an equivalence class of size at least n. Then there is $\mathcal{M} \models T$ with an infinite equivalence class. Conclude that the class of all equivalence relations where every class is finite is not an elementary class.

2) Let $\mathcal{L} = \{R\}$ where R is a binary relation. Recall that a *graph* is an $\mathcal{L}$-structure $\mathcal{M}$ where $R^{\mathcal{M}}$ is symmetric and irreflexive. We say that a graph is *connected* if for each $x \neq y$ we can find a path from x to y. Prove that the class of connected graphs is not an elementary class.

3) Let $\mathcal{L}$ be the language $\{\cdot, e\}$ where $\cdot$ is a binary relation and e is a constant symbol. Let $\mathcal{K}$ be the class of all groups where every element has finite order. Prove that $\mathcal{K}$ is not an elementary class.

4) Let $\mathcal{L}$ be the language with one binary relation symbol $<$. Let T be an $\mathcal{L}$-theory extending the theory of linear orders such that T has infinite models. Show that there is $\mathcal{M} \models T$ and an order preserving embedding $\sigma : \mathbb{Q} \to M$ of the rational numbers into M. For example there is $\mathcal{M} \models \mathrm{Th}(\mathbb{Z}, <)$ in which the rational order embedds.[Hint: Add constants c_q for all $q \in \mathbb{Q}$ and let $T^* = T \cup \{c_q < c_r : q, r \in \mathbb{Q}, q < r\}$.]

5) We say that an ordered abelian group $(G, +, 0, <)$ is *archimedian* if for all $x, y \in G$ with $x, y > 0$ there is a positive integer m such that $x < my$. Show that there are non-archimedian models of $\mathrm{Th}(\mathbb{Q}, +, <)$.

6) Let $\mathcal{L} = \{s\}$, where s is a unary function symbol. Let T be the $\mathcal{L}$-theory that asserts that s is a bijection with no cycles (i.e., $s^{(n)}(x) \neq x$ for $n = 1, 2, \ldots$). For which cardinals κ is T κ-categorical? Is T complete?